The difference calculus for functors on presheaves

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Abstract.

We start with the one-variable case, developing a difference calculus for endofunctors F of **Set**. Like in the classical difference calculus for functions $\mathbb{R} \to \mathbb{R}$, a discrete version of the derivative, we define $\Delta[F] \colon \mathbf{Set} \to \mathbf{Set}$ by

$$\Delta[F](X) = F(X+1) \setminus F(X).$$

The "$\setminus$" is set difference so $\Delta[F]$ shouldn't be expected to be functorial, but it is for a rather large class of functors, the taut functors of Manes. (A functor is taut if it preserves inverse images or, put differently, preserves pullbacks along monos.)

We develop the difference calculus for these, obtaining limit and colimit rules analogous to the classical product and sum rules. We get a lax chain rule where none exists for mere functions, and a Newton summation formula which appears as a left adjoint. Many interesting classes of functors are taut, polynomial and analytic ones for example, and for these we give explicit descriptions of their differences.

We then proceed to the multivariable case, i.e. functors between presheaf categories. The generalization of tautness we need is the preservation of complemented subobjects and their inverse images. We then get the partial differences $\Delta_A[F]$ by replacing the 1 in the definition of $\Delta[F]$ by the representable at A. All the Δ_A together form a profunctor, $\nabla[F]$, the Jacobian of F. We then establish similar rules as in the one-variable case and study some examples.