When limits are limits: Topological enrichment with an application to probability

P. Perrone

Paolo Perrone (www.paoloperrone.org)
University of Oxford

Ruben Van Belle (ruben.van.belle@cs.ox.ac.uk)
University of Oxford

Abstract.

Sequential limits and colimits are often used, intuitively, to "approximate spaces" from below or from above. For example, the diagram in Set formed by finite sets and inclusion maps

$$\{0\} \hookrightarrow \{0,1\} \hookrightarrow \{0,1,2\} \hookrightarrow \{0,1,2,3\} \hookrightarrow \dots$$

has as colimit the set of natural numbers $\mathbb{N}$. The same is true more generally for filtered colimits and cofiltered limits. These examples are probably the motivation for the name "limit", by analogy with topological limits, which also "approximate" things, usually numbers or points in a space.

When we have a filtered or cofiltered diagram of subobjects $(A_\lambda)_{\lambda\in\Lambda}$ of a given object X and inclusion maps, the cofiltered limit gives the infimum ("intersection") of subspaces, and the filtered colimit gives the supremum ("union"). Whenever these subspaces are retracts $(\iota : A \to X, \pi : X \to A)$, they give rise to idempotent morphisms $e = \iota \circ \pi : X \to X$, and if the category is Cauchy-complete, every idempotent arises in this way.

Using the canonical closed monoidal structure of Top, one can consider topologically enriched categories. In such an enriched category C, one can look if whenever a net of retracts $A_\lambda \subseteq X$ tends to a retract A as a *limit* (= infimum) or *colimit* (= supremum), the corresponding idempotents e_λ tend to e *topologically*, in the hom-space $\mathrm{C}(X, X)$. We call these properties, which may hold or fail depending on C, the *upward* and *downward Levi properties*, respectively for suprema and infima, in analogy with Beppo Levi's theorem, which says that every bounded monotone real sequence converges to its supremum.

An example of topologically enriched category where these properties hold is the category of Hilbert spaces and short maps [4, 5]. In this case, every (split) idempotent is the projection onto a closed subspace. Cofiltered limits of subspaces and inclusions give exactly intersections, and filtered colimits give closures of unions. A net of projectors (e_λ) onto subspaces A_λ tends topologically to a projector e if and only if the A_λ tend, as limit or colimit, to the subspace A splitting the idempotent e.

Another such category arises in probability theory, the category of standard Borel probability spaces and couplings between them [2]. Retracts, in this category, are sub-sigma-algebras up to almost sure equality. The upward and downward Levi properties hold, and give exactly convergence in mean for martingales and inverse martingales, cornerstone results of probability theory, which now have a categorical formalization and proof.

References

[1] Paolo Perrone and Ruben Van Belle, *Convergence of martingales via enriched dagger categories.* In preparation.

[2] Noé Ensarguet and Paolo Perrone, *Categorical probability spaces, ergodic decompositions, and transitions to equilibrium.* Submitted (2023), available at arXiv:2310.04267.

[3] Ruben Van Belle, *A categorical treatment of the Radon-Nikodym theorem and martingales.* Submitted (2023), available at arXiv:2305.03421.

[4] Chris Heunen and André Kornell, *Axioms for the category of Hilbert spaces.* Proceedings of the National Academy of Sciences USA 119(9), 2022.

[5] Matthew Di Meglio and Chris Heunen, *Dagger categories and the complex numbers: Axioms for the category of finite-dimensional Hilbert spaces and linear contractions.* Submitted (2024), available at arXiv:2401.06584.