A new centre for crossed modules

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Abstract.

Crossed modules are algebraic models of homotopy 2-types. By definition, a crossed module G_* is a group homomorphism $\partial : G_1 \to G_0$ together with an action of G_0 on G_1 satisfying some properties. The most important invariants of the crossed module G_* are the group $\pi_0(G_*) = Coker(\partial)$ and the $\pi_0(G_*)$-module $\pi_1(G_*) = Ker(\partial)$.

One of the main results of this paper is to show that any crossed module $\partial : G_1 \to G_0$ fits in a commutative diagram

$$\begin{array}{ccc} G_1 & \xrightarrow{\delta} & \mathbf{Z}_0(G_*) \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \mathsf{z}_0} \\ G_1 & \xrightarrow{\partial} & G_0 \end{array}$$

where the top horizontal $G_1 \xrightarrow{\delta} \mathbf{Z}_0(G_*)$ and right vertical $\mathbf{Z}_0(G_*) \xrightarrow{\mathsf{z}_0} G_0$ arrows have again crossed module structures. In fact, the first one is even a braided crossed module, which we call the *centre of the crossed module* $\partial : G_1 \to G_0$ and denote by $\mathbf{Z}_*(G_*)$.

We show that the braided monoidal category corresponding to the braided crossed module $\mathbf{Z}_*(G_*)$ is isomorphic to the Drinfeld centre of the monoidal category corresponding to G_*.

Our definition of $\mathbf{Z_0}(G_*)$ is based on certain crossed homomorphisms $G_0 \to G_1$ and has some advantage compared to one based on monoidal categories. Namely, the description of $\mathbf{Z}_*(G_*)$ in terms of crossed homomorphisms makes it easy to relate the centre of a crossed module to group cohomology. The essential invariants of $\mathbf{Z}_0(G_*)$ are closely related to low dimensional group cohomology. In fact, one has an isomorphism of groups $\pi_1(\mathbf{Z}_*(G_*)) \cong H^0(\pi_0(G_*), \pi_1(G_*))$ and the group $\pi_0(\mathbf{Z}_*(G_*))$ fits in an exact sequence

$$0 \to H^1(\pi_0(G_0), \pi_1(G_*)) \to \pi_0(\mathbf{Z}_*(G_*)) \to \mathsf{Z}_{\pi_1(G_*)}(\pi_0(G_*)) \to H^2(G_0, \pi_1(G_*)),$$

where $\mathsf{Z}_{\pi_1(G_*)}(\pi_0(G_*))$ is the subgroup of the centre of the group $\pi_0(G_*)$ consisting of those elements which act trivially on $\pi_1(G_*)$.

It should be pointed out that in the 80's Norrie also introduced the notion of a centre of a crossed module, but our notion differs from hers. Our centre can be shown to be a homotopy invariant, unlike hers.

I'll also briefly discuss a connection between this definition and the Gottlieb group of the classifying space of the crossed module.