The arrows between double category sites for Grothendieck topoi

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Abstract In [3] Ehresmann sites are introduced as a way to represent étendues by sites where the underlying categories are ordered groupoids. This result was proved through a correspondence between Ehresmann sites and left cancellative categories (which represent étendues, [2]). In [1], the notion of ordered groupoid was reinterpreted as a special type of double category: an internal groupoid in the category of posets, with the additional property that the source map from the groupoid of arrows to the groupoid of objects is a discrete fibration. The Ehresmann topology is defined on the vertical arrows but the horizontal arrows contribute to the conditions the covering families need to satisfy. This reinterpretation of Ehresmann sites gives rise to a *2-category* of Ehresmann sites such that the correspondence between left cancellative sites and Ehresmann sites given in [3] becomes part of:

- a 2-adjunction between the 2-category of left cancellative categories and the 2-category of ordered groupoids;
- a biequivalence between the 2-category of left cancellative categories and the 2-category of ordered groupoids where each connected component of the vertical category has a maximal object;
- a biequivalence between the 2-category of left cancellative Grothendieck sites and the 2-category of Ehresmann sites.

This last equivalence is obtained by characterizing the functors between Ehresmann sites that correspond to cover-preserving, covering-flat functors between Grothendieck sites and restricting the previous equivalence.

In our current work we introduce generalized Ehresmann sites that represent arbitrary Grothendieck topoi. Julia Ramos González will introduce the definition of generalized Ehresmann site in her talk and describe the type of Grothendieck sites we use to obtain an equivalence between the 2-categories of sites. The Grothendieck sites come with a left quadrable orthogonal factorization system where the

left class of arrows are covering in the Grothendieck topology and the right arrows are monic (we do not require the left class to contain all coverings and neither the right class to contain all monics).

I will present the generalization of the 2-adjunction just listed for left cancellative categories and Ehresmann sites to a 2-adjunction between the 2-category of categories with an orthogonal factorization system where the right class contains only monic arrows and the 2-category of ordered categories (double categories where the vertical structure is posetal and the domain functor is a discrete fibration). This 2-adjunction restricts to a biequivalence when we restrict ourselves to ordered categories where each component of the vertical category has a maximal object. (This result closely resembles the types of correspondences given in [4] but the way we define the double category from a factorization system is different: rather than taking the monic arrows themselves as vertical arrows, we take the subobjects they define.)

This gives us a good starting point to define the arrows between the generalized Ehresmann sites described by Julia in her talk: we have a correspondence on objects and we use cover-preserving, covering-flat arrows that preserve the factorization systems as arrows between the Grothendieck sites. The latter property assures us that we have corresponding double functors. So our job is now to translate the notions of being cover preserving and covering flat to maps between generalized Ehresmann sites. It is obvious what it means to be covering preserving, but the notion of covering flatness requires further work: which finite diagrams for an Ehresmann site should be used to test for covering-flatness? We need that cones over such finite diagrams for generalized Ehresmann sites correspond in a suitable fashion to cones for finite diagrams for our chosen Grothendieck sites.

To resolve this issue we show that each finite diagram $D\colon \mathcal{I} \to (\mathfrak{a}, J_{\mathfrak{a}})$ into a Grothendieck site with a suitable orthogonal factorization system factors through an indexing category $\widetilde{\mathcal{I}}$ with a strict factorization system so that the induced diagram is a functor of categories with factorization systems and such that cones for the first extend uniquely to cones for the second. The first correspondence in [4], for strict factorization systems, gives then the corresponding indexing double category, that gives us a finite diagram in the corresponding generalized Ehresmann site. This construction allows us to define covering flatness for maps between generalized Ehresmann sites in terms of finite diagrams indexed by finite double categories where the domain functor is a discrete fibration.

We now obtain a 2-category of generalized Ehresmann sites and covering-preserving covering-flat double functors (and Λ-transformations that form a straight generalization of those introduced in [1]). This 2-category is biequivalent to the 2-category of our chosen Grothendieck sites.

Finally, Comparison Lemma maps between our Grothendieck sites satisfy the conditions to form a bicategory of fractions. This allows us to obtain a class of Comparison Lemma maps between generalized Ehresmann sites so that Grothendieck topoi form the bicategory of fractions for generalized Ehresmann sites with respect to these morphisms.

References

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