Double categorical presentations of Grothendieck topoi

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Abstract. The orthogonal epi-mono factorization system in a Grothendieck topos together with the stability under pullbacks of epimorphisms (and monomorphisms) allows to fully encode the canonical topology on the topos, given by the jointly-epimorphic families, in terms of:

(M) the families of monic arrows that are jointly-epimorphic (i.e. the families of monic arrows that are covering);

(C) the epimorphic arrows (i.e. the single arrows that are covering);

(S) the stability under pullbacks of epimorphisms (and monomorphisms) along both epimorphisms and monomorphisms (i.e. single covering arrows and monic arrows satisfy the Ore condition).

On one hand, **(M)** points us towards the direction of left cancellative sites, i.e. sites where all morphisms are monic, and their categories of sheaves, the étendues. On the other hand, **(C)** + **(S)** point us towards the sites where all single arrows are covering, the atomic sites, and their categories of sheaves, the atomic topoi.

In this talk, inspired by this observation, we introduce the notion of *covering-mono Grothendieck site*, a small Grothendieck site endowed with an orthogonal factorization system in which the left class consists of single covering arrows and the right class consists of monomorphisms, together with suitable Ore conditions. We define *covering-mono morphisms of sites* to be the covering-flat covering-preserving morphisms that also preserve the factorization system. We then show that every Grothendieck topos admits a covering-mono site presentation and that the 2-category of Grothendieck topoi can be recovered as a bicategory of fractions of the 2-category of covering-mono sites, where one inverts the covering-mono morphisms that are LC (Lemme de Comparaison) [2]. Every Grothendieck topos can be seen in this way as an interpolation between an étendue and an atomic topos.

Étendues can also be presented in terms of Ehresmann sites [3], which are ordered groupoids endowed with an Ehresmann topology and that can be envisioned as suitable double categories where the Ehresmann topology lives in the vertical direction [1]. In the same way that covering-mono sites (presenting Grothendieck topoi) generalize left cancellative sites (presenting étendues), we introduce the

notion of *generalized Ehresmann site* (generalizing classical Ehresmann sites in their double categorical incarnation) and we show that every Grothendieck topos can be presented in terms of a generalized Ehresmann site. More concretely, we define a generalized Ehresmann site as a category internal in posets (which we envision as a double category), with suitable horizontal and 2-cellular Ore conditions and endowed vertically with an Ehresmann topology. The horizontal arrows are now not isomorphisms in general, as it was the case for the classical Ehresmann sites, but the Ore conditions imposed allow us to treat them as single covering arrows. In addition, we introduce the appropriate notion of sheaf on a generalized Ehresmann site, analogous to the notion of sheaf on a classical Ehresmann site. We then establish a direct connection between covering-mono Grothendieck sites and generalized Ehresmann sites enhancing the connection between left cancellative sites and classical Ehresmann sites from [3] and [1]: to each covering-mono Grothendieck site we associate a generalized Ehresmann site and vice versa, and we show that these associations respect the operation of taking sheaves. This allows us to conclude that generalized Ehresmann sites provide presentations for Grothendieck topoi, as desired.

We will present the relation between covering-mono Grothendieck sites and generalized Ehresmann sites exclusively at the level of objects. However, this relation can be understood at the level of bicategories (as it is the case for left cancellative sites and classical Ehresmann sites, see [3] and [1]). The relation at the bicategorical level and the recovery of the 2-category of Grothendieck topoi as a bicategory of fractions of the 2-category of generalized Ehresmann sites by inverting the corresponding class of LC morphisms of generalized Ehresmann sites will not be treated in this talk, but will be presented by Dorette Pronk in hers.

While LC morphisms of sites (resp. LC covering-mono morphisms of sites) admit a calculus of fractions [4] allowing to recover the 2-category of Grothendieck topoi as a bicategory of fractions of the 2-category of sites (resp. of the 2-category of covering-mono sites), the 2-categories of left cancellative sites and classical Ehresmann sites are too restrictive in order for the LC morphisms to admit a calculus of fractions. In the last part of the talk, we identify larger families of presentations of étendues that solve this issue. More concretely, we identify a subclass of the covering-mono sites, the *torsion-free generated covering-mono sites*, enlarging the subclass of left cancellative sites but still providing presentations of étendues. In parallel, we describe the corresponding subclass of the generalized Ehresmann sites, which we call the *torsion-free generated generalized Ehresmann sites*. We show that the LC morphisms in the class of torsion-free generated covering-mono sites do admit a calculus of fractions allowing to recover the 2-category of étendues as a bicategory of fractions. Through the bicategorical relation between covering-mono sites and generalized Ehresmann sites that Dorette Pronk will present in her talk, it is immediate to obtain the analogous result for the torsion-free generated generalized Ehresmann sites.

References

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