Lawvere Theories and Symmetric Operads as Substitution Algebras: Free constructions for Abstract Syntax

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Abstract. We are interested in free constructions of Lawvere theories and of symmetric operads from syntactic structures. These respectively correspond to the syntax of cartesian and symmetric monoidal second-order theories, and most generally arise from binding signatures. For a direct constructive approach to these free constructions, it is best to consider Lawvere theories and symmetric operads as *substitution algebras* modelling single-variable substitution.

In the cartesian case, one considers the object-classifier topos $\mathcal{F} = \mathbf{Set}^{\mathbb{F}}$ (where $\mathbb{F}$ is the category of finite cardinals and functions) as the ambient category for models. Here, a substitution algebra [4] consists of a presheaf, $X \in \mathcal{F}$, a variable operation $\nu : 1 \to \delta(X)$, and a single-variable substitution operation $\sigma : \delta(X) \times X \to X$ satisfying the following four axioms:

$$\begin{tikzcd} 1 \times X \arrow[r, "\pi_2"] \arrow[d, "\nu\times\mathrm{id}"'] & X \\ \Sigma_{\mathrm{s}}(X) \arrow[ur, "\sigma"'] & \end{tikzcd} \qquad \begin{tikzcd} X \times X \arrow[r, "\pi_1"] \arrow[d, "\mathrm{up}_X\times\mathrm{id}"'] & X \\ \Sigma_{\mathrm{s}}(X) \arrow[ur, "\sigma"'] & \end{tikzcd} \qquad \begin{tikzcd} \delta^2(X) \times 1 \arrow[rr, "\pi_1"] \arrow[d, "\mathrm{id}\times\nu"'] & & \delta^2(X) \arrow[d, "\mathrm{cont}"] \\ \delta^2(X) \times \delta(X) \arrow[r, "\cong"] & \delta\Sigma_{\mathrm{s}}(X) \arrow[r, "\delta(\sigma)"] & \delta(X) \end{tikzcd}$$

$$\begin{tikzcd} \delta\Sigma_{\mathrm{s}}(X) \times X \arrow[r, "\cong"] \arrow[d, "\delta(\sigma)\times\mathrm{id}"'] & \Sigma_{\mathrm{s}}\delta(X) \times X \arrow[r, "\mathbf{str}"] & \Sigma_{\mathrm{s}}\Sigma_{\mathrm{s}}(X) \arrow[r, "\Sigma_{\mathrm{s}}(\sigma)"] & \Sigma_{\mathrm{s}}(X) \arrow[d, "\sigma"] \\ \Sigma_{\mathrm{s}}(X) \arrow[rrr, "\sigma"] & & & X \end{tikzcd}$$

In the above definition, $(\delta, \mathrm{up}, \mathrm{cont}, \mathrm{swap})$ is a *symmetric monad* on $\mathcal{F}$, induced by the structure of $\mathbb{F}$, and $\Sigma_{\mathrm{s}}(A) = \delta(A) \times A$ is a strong endofunctor on $\mathcal{F}$. Substitution algebras are equivalent to Lawvere theories [4] and provide a finite equational presentation of Lawvere theories over $\mathcal{F}$ [5], in contrast to their countably-sorted presentation as abstract clones.

Furthermore, $\mathcal{F}$ is the suitable environment for second-order cartesian theories, conservatively extending Lawvere theories [1]. Binding signatures account for algebraic operations with variable binding; they appear, for example, as abstraction in the lambda calculus and quantifiers in predicate logic [3]. An endofunctor Σ on $\mathcal{F}$ constructed using the product, coproduct, and δ may be associated with each binding signature. We prove that the free Σ-algebra over the presheaf of abstract variables, $\mathcal{Y}(1) = \mathbb{F}(1, -) : \mathbb{F} \hookrightarrow \mathbf{Set}$, is equipped with a canonical substitution algebra structure which is induced by *generalised parametrised structural recursion.* This structure models the abstract syntax of the binding signature and is initial in the category of Σ-algebras with compatible substitution algebra structure.

For symmetric monoidal theories – for which the first-order theories are symmetric operads – the appropriate ambient category for models is that of species of structures, $\mathcal{B} = \mathbf{Set}^{\mathbb{B}}$, where $\mathbb{B}$ is the groupoid of finite cardinals [6, 7]. $\mathcal{B}$ has an additional monoidal tensor, namely the Day tensor product, $\otimes$. This is used, instead of the cartesian product, to model linear pairing. The analogous δ on $\mathcal{B}$ is only a *symmetric endofunctor* and does not respect linear pairings as in the cartesian case. Instead, it is a derivative operator, equipped with a *Leibniz* canonical natural isomorphism, $\delta(A) \otimes B + A \otimes \delta(B) \xrightarrow{\cong} \delta(A \otimes B)$.

An endofunctor on $\mathcal{B}$ for a binding signature Σ is constructed using the Day tensor, coproduct, and the derivative δ. Using the Leibniz isomorphism, we define a *derived* functor $\Sigma' : \mathcal{B}^2 \to \mathcal{B}$ together with a canonical isomorphism $\delta\Sigma(A) \cong \Sigma'(A, \delta(A))$.

We define a linear substitution algebra (equivalent to that of [2]) as a presheaf $Y \in \mathcal{B}$ together with a variable operation $v : I \to \delta(Y)$ and a single-variable substitution operation $\varsigma : \delta(Y) \otimes Y \to Y$ satisfying the following two axioms:

$$\begin{tikzcd} I \otimes Y \arrow[r, "\cong"] \arrow[d, "v\otimes\mathrm{id}"'] & Y \\ \delta(Y) \otimes Y \arrow[ur, "\varsigma"'] & \end{tikzcd} \qquad \begin{tikzcd} \delta\Sigma_{\mathrm{s}}(Y) \otimes Y \arrow[r, "\cong"] \arrow[d, "\delta(\varsigma)\otimes\mathrm{id}"'] & \Sigma'_{\mathrm{s}}(Y, \delta(Y)) \otimes Y \arrow[r, "\mathbf{str}_{\mathrm{s}}"] & \Sigma'_{\mathrm{s}}(Y, \delta(Y) \otimes Y) \arrow[r, "\Sigma'_{\mathrm{s}}(\mathrm{id},\varsigma)"] & \Sigma'_{\mathrm{s}}(Y, Y) \arrow[d, "\varsigma+\varsigma"] \\ \delta(Y) \otimes Y \arrow[rrr, "\varsigma"] & & & Y \end{tikzcd}$$

Here, $\Sigma_{\mathrm{s}}(A) = \delta(A) \otimes A$ and $\Sigma'_{\mathrm{s}}(A, B) = \delta(B) \otimes A + \delta(A) \otimes B$. The category of linear substitution algebras is equivalent to the category of symmetric operads (and, indeed, the category of simultaneous-substitution monoids [8]).

We prove that the free Σ-algebra over $\mathcal{Y}(1) = \mathbb{B}(1, -) : \mathbb{B} \to \mathbf{Set}$ has an induced linear substitution algebra that is initial in the category of Σ-algebras with compatible linear substitution algebra structure. The full study of second-order symmetric monoidal theories remains work in progress.

References

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