Cotraces and Inner Product Enrichment of Bicategories

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Abstract. The worlds of category theory and linear algebra seem to be closely, but informally, interwoven. On the one hand, many category theoretic contructions have strong analogues in linear algebra. Most obvious, perhaps, is the similarity between categorical and linear adjoints. But it is often useful to find other comparisons as well: categorical limits might be compared to Cauchy limits, or products; profunctors might be compared to matrices; the Yoneda lemma might be compared to the Riesz representation theorem.

On the other hand, category theory has been proven to be a useful tool in the abstraction of linear algebraic results. Compact closed categories, dagger categories, scalars and traces in monoidal categories all provide valuable abstractions that allow us to do away with matrix computation and work much more intuitively with string diagrams. For the particular case of dagger compact categories see, for example, work of Doplicher and Roberts [1], Baez and Dolan [2], Abramsky and Coecke [3], Selinger [4], and Heunen and Kornell [5].

We are interested here in the varied and subtle roles that scalars, traces and inner products play in the world of *bi*categories. In particular, trace-like structures are well-studied. Thinking of bicategories as generalised monoidal categories, Ponto [6] gave the structure and conditions necessary to take the trace of an endo-2-cell, and this formalism has been used for a number of applications in topology. Ponto and Shulman [7] later studied the various category theoretic properties of this notion of trace. Much more recently, Hess and Rasekh [8] related this form of trace to topological Hochschild homology. Work by Bartlett [9], and Ganter and Kapranov [10], gave the definition of the 'categorical trace' or '2-trace' which is defined by considering the bicategory of 2-Vector spaces. And as far back as 1997, Day and Street [11] published an account of compact closed bicategories in which they gave the definition of a *co*trace.

Our new work shows that this cotrace has an incredibly rich structure and enjoys a number of trace-like properties. What's more, it can be used to define a sort of categorical inner product – defined analogously to the Frobenius inner product – which gives an enrichment to the whole bicategory. In the same way that the Frobenius inner product is a scalar that exists 'between' two linear maps, this scalar enrichment replaces every set of 2-cells with a categorical scalar.

This result has several consequences. Firstly, it highlights a canonical enrichment that gives many well-known bicategories their extra structure. For example, it allows us to replace sets of enriched natural transformations with their corresponding natural transformation object.

Secondly, it unifies Day and Street's cotrace with the 'categorical trace' as defined by Ganter and Kapranov and the '2-trace' as defined by Bartlett for 2-Hilbert spaces. It turns out that the 2-trace is simply the unenriched cotrace.

Thirdly, it provides a theoretical underpinning for Willerton's [12] observation that the 2-trace seems to be somehow dual to the usual notion of trace. Willerton pointed out that, if we extend the

definitions of trace and 2-trace to the context of a bicategory with duals, these two different traces often appear to give opposing results. For example, in the bicategory of profunctors the trace gives a coend, but the 2-trace gives an end; in a particular bicategory of bimodules, the trace gives Hochschild homology, but the 2-trace gives Hochschild cohomology. The problem with this observation was that the trace is a scalar – that is, a map from the unit object to itself – whereas the 2-trace is a *set* of 2-cells. It is only after adding appropriate enrichment that this ostensible duality makes sense.

Finally, it is a further step towards formalising the relationship between categorical adjoints and adjoints of linear maps. Since the enrichment is defined, and behaves, like an inner product, 1-cells that are adjoint in the category theoretic sense are also adjoint in a linear sense.

In this talk we will explore how the enrichment is defined, the similarities that exist between the enrichment and the Frobenius inner product, the trace-like properties that this endows the cotrace with, and the many bicategories for which the trace and cotrace give interesting and dual constructions.

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