Adjoint split extensions of categories

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Abstract.

Suppose we are given a parameterized monad, meaning a functor $\mathcal{B} \to \mathsf{Mnd}(\mathcal{X})$ from $\mathcal{B}$ to the category of monads on $\mathcal{X}$. This can be interpreted as a particular kind of action of the category $\mathcal{B}$ on the category $\mathcal{X}$. The data of the functor $\mathcal{B} \to \mathsf{Mnd}(\mathcal{X})$ can also be given as a functor

$$\mathcal{B} \times \mathcal{X} \to \mathcal{X}\,, \quad (B, X) \mapsto B \cdot X\,,$$

where each endofunctor $B \cdot -$ carries the structure of a monad.

To each monad $B \cdot -$ we can associate its category of algebras, and when we glue these categories together using the Grothendieck construction, we get the category $\mathcal{X} \rtimes \mathcal{B}$, whose objects are triples

$$(X, B, B \cdot X \xrightarrow{\xi} X)\,,$$

with $X \in \mathsf{Ob}(\mathcal{X})$, $B \in \mathsf{Ob}(\mathcal{B})$ and ξ a $(B \cdot -)$-algebra on X. We treat an algebra $B \cdot X \xrightarrow{\xi} X$ as the data of an action the object B on the object X, so $\mathcal{X} \rtimes \mathcal{B}$ can be considered as a category of actions, with the parameterized monad $\mathcal{B} \to \mathsf{Mnd}(\mathcal{X})$ specifying what it means for $B \in \mathsf{Ob}(\mathcal{B})$ to act on $X \in \mathsf{Ob}(\mathcal{X})$.

The category $\mathcal{X} \rtimes \mathcal{B}$ has an associated fibration

$$p\colon \mathcal{X} \rtimes \mathcal{B} \to \mathcal{B}\,, \quad (X, B, \xi) \mapsto B$$

and, as long as $\mathcal{X}$ and $\mathcal{B}$ have initial and terminal objects and the monad $\mathbf{0} \cdot -$ corresponding to the initial object $\mathbf{0} \in \mathsf{Ob}(\mathcal{B})$ is the identity monad, we can form the diagram

$$\mathcal{X} \underset{i^R}{\overset{i}{\underset{\perp}{\rightleftarrows}}} \mathcal{X} \rtimes \mathcal{B} \;\; \begin{matrix} \xleftarrow{\;s\;} \\ \perp \\ \xrightarrow{\;p\;} \\ \perp \\ \xleftarrow[\;p^R\;]{} \end{matrix} \;\; \mathcal{B}$$

of categories and adjunctions. We will argue that this diagram is an **adjoint split extension** in a suitable setting, with the word *adjoint* indicating that the splitting s of p is left adjoint to p.

In fact, we will treat this diagram as the archetypal adjoint split extension, and describe a theory of such extensions, where the $\mathcal{X} \rtimes \mathcal{B}$ construction will play the role of the semi-direct product that we construct from the action $\mathcal{B} \to \mathsf{Mnd}(\mathcal{X})$. We view these extension in the setting where the morphisms between categories are adjunctions, meaning we need to specify what *exactness* means for a sequence

$$\mathcal{X} \underset{i^R}{\overset{i}{\underset{\bot}{\rightleftarrows}}} \mathcal{A} \underset{p^R}{\overset{p}{\underset{\bot}{\rightleftarrows}}} \mathcal{B}$$

whose composite is the zero adjunction, meaning the adjunction between the constantly initial and constantly terminal functors. This exactness turns out to be approximately equivalent to saying that the full subcategories $\mathcal{X}$ and $\mathcal{B}$ form a torsion theory in $\mathcal{A}$.

For any adjoint split extension

$$\mathcal{X} \underset{i^R}{\overset{i}{\underset{\bot}{\rightleftarrows}}} \mathcal{A} \; \begin{smallmatrix} \xleftarrow{\;s\;} \\ \bot \\ \xrightarrow{\;p\;} \\ \bot \\ \xleftarrow[\;p^R\;]{} \end{smallmatrix} \; \mathcal{B}$$

we have a comparison $\mathcal{A} \to \mathcal{X} \rtimes \mathcal{B}$, which, under suitable extra assumptions on the diagram, is an equivalence. For example, we get $\mathcal{A} \to \mathcal{X} \rtimes \mathcal{B}$ when our categories are the opposites of toposes, as observed in [3], in which case $\mathcal{X} \rtimes \mathcal{B}$ is just Artin gluing. We can also show that $\mathcal{A} \simeq \mathcal{X} \rtimes \mathcal{B}$ when we are working with semi-abelian categories, and, for example, [4] can be seen as making use of the equivalence $\mathsf{ccHopf}_{\mathbb{K}} \simeq \mathsf{Lie}_{\mathbb{K}} \rtimes \mathsf{Group}$, where $\mathsf{ccHopf}_{\mathbb{K}}$ is the category of cocommutative Hopf algebras over a nice field $\mathbb{K}$.

In categorical algebra, one often encounters the functor

$$\flat\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}\,, \quad (B, X) \mapsto B \flat X := \ker(X + B \xrightarrow{[0,1]} B)$$

for a pointed category $\mathcal{A}$ with coproducts and pullbacks, and from our point of view this functor is the action associated to the adjoint split extension of points

$$\mathcal{A} \underset{i^R}{\overset{i}{\underset{\bot}{\rightleftarrows}}} \mathsf{Pt}_{\mathcal{A}} \; \begin{smallmatrix} \xleftarrow{\;s=p^R\;} \\ \bot \\ \xrightarrow{\;p\;} \\ \bot \\ \xleftarrow[\;p^R\;]{} \end{smallmatrix} \; \mathcal{A}\,.$$

The properties of the comparison $\mathsf{Pt}_{\mathcal{A}} \to \mathcal{A} \rtimes \mathcal{A}$ can be used to characterize the properties of $\mathcal{A}$. For example, conservativity of the comparison can be used to characterize the *protomodularity* of $\mathcal{A}$, and being an equivalence means that $\mathcal{A}$ is a *category with semidirect products* in the sense of [2].

References

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