Bigraded path homology
and the magnitude-path spectral sequence

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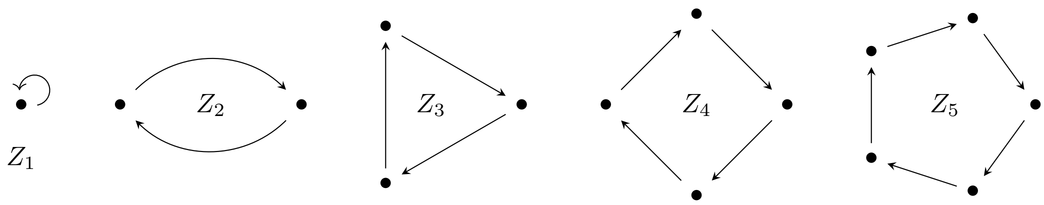
Abstract. The past two decades have seen a proliferation of homology theories for graphs (directed and undirected), including discrete and cubical homology, path homology, magnitude homology and reachability homology. Each of these theories is homotopy-invariant in an appropriate sense and satisfies some sensible discrete analogue of the Eilenberg–Steenrod axioms. Despite this, they tend to disagree even on quite primitive classes of graphs. For instance, magnitude homology distinguishes the directed cycles Z_m for every $m \geq 1$; path homology sees Z_1 and Z_2 as 'contractible' and all the rest as 'circle-like'; and to reachability homology every directed cycle appears contractible.



Thus, the evolving story of the homology of graphs is not a simple retelling of the classical story for spaces (in which Eilenberg–Steenrod's axiomatization guarantees a theory that is essentially unique). This talk, based on [3] and [4], presents a new chapter in the tale.

The diversity of homological viewpoints has motivated a recent drive towards consolidation using the framework of formal homotopy theory. Results so far have mainly been negative: for various natural notions of weak equivalence of graphs, it is known that no model structure can exist. But there has been one positive result: Carranza *et al* [2] exhibit a cofibration category structure on the category of directed graphs, for which the weak equivalences are maps inducing isomorphisms on path homology. One of the interesting features of their work is the choice of cofibrations, whose definition is reminiscent of the pairs of spaces for which the Mayer–Vietoris theorem holds in magnitude homology. As it turns out, this is no coincidence. Asao has shown that the two homology theories are closely related, appearing on consecutive pages in a certain spectral sequence [1].

The talk will describe ongoing work to understand that sequence, now known as the *magnitude-path spectral sequence* (or MPSS) of a directed graph. We refer to the underlying filtered chain complex as the *reachability complex*; its homology—the target object of the MPSS—is reachability homology.

Page 1 of the MPSS is exactly magnitude homology, while page 2 is a natural refinement of path homology, which lies its horizontal axis; we call this page the *bigraded path homology* of a directed graph. The sequence thus encompasses several existing invariants, and clarifies the relationships between them, while adding infinitely many new ones. The invariance properties of the pages grow progressively stronger as one passes through the sequence, giving rise to a nested family of weak equivalence classes of directed graphs. For instance, page r of the MPSS sees the directed m-cycle Z_m as contractible when m is less than r, and distinguishes each of the Z_ms for $m \geq r$.

Concerning the spectral sequence as a whole, our main results are as follows.

Theorems *Every page of the MPSS has the following homological properties:*

- *It satisfies an excision theorem with respect to the cofibrations in* [2].
- *It satisfies a Künneth theorem with respect to the box product.*
- *It is a finitary functor on the category of directed graphs (meaning it preserves filtered colimits).*

In particular these hold for bigraded path homology, which also satisfies a Mayer–Vietoris theorem.

This allows us to show that the cofibration category structure in [2] admits a natural refinement.

Theorem *The category of directed graphs carries a cofibration category structure in which the cofibrations are those of* [2] *and the weak equivalences are maps inducing isomorphisms on bigraded path homology. This is a strictly finer structure than the one exhibited in* [2]*: for instance, bigraded path homology, unlike ordinary path homology, distinguishes the directed m-cycles for every $m \geq 2$.*

These results have consequences of three sorts. Firstly, they demonstrate the value of bigraded path homology as a novel invariant of directed graphs, sharing the good properties of ordinary path homology, but with greater distinguishing power. Thus, in applications where path homology might be used, it is worth considering the bigraded variant.

Secondly, on a technical front, our methods illustrate a useful principle: that properties of path homology are frequently (though not always) 'inherited' from corresponding properties of magnitude homology—and that what holds true for either of these will often hold true throughout the MPSS. Moreover, arguments at the level of the reachability complex tend to be more straightforward than the rather involved proofs necessitated by the standard construction of path homology. Thus, when trying to prove statements about path homology, it is worth considering whether they can be approached via the spectral sequence.

Finally, we hope the MPSS will eventually cast more light on the homotopy theory of directed graphs. It is tempting to speculate that the cofibration category we describe may belong to a nested family of structures, one for each page; time permitting, we will sketch this idea at the end of the talk.

References

[1] Y. Asao, *Magnitude homology and path homology*, Bull. Lond. Math. Soc. 55 (2023), no. 1, 375–398.

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[3] R. Hepworth and E. Roff, The reachability homology of a directed graph, preprint arXiv:2312.01378, 2023.

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