Ultracompletions

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Abstract. The notion of ultracategory was introduced by Michael Makkai in [8] for the characterisation of categories of models of pretoposes, an ample extension to (intuitionistic) first order theories of Stone duality for Boolean algebras, providing a kind of Stone duality for first order theories—aka cooneptual completeness. Recently, Jacob Lurie refined that notion in [7] producing another approach to the duality for pretoposes—the two notions of ultracategory appear to be different, though no separating example has been produced yet. All this suggests that there are already two forms of duality for first order theories, in line with Esakia's duality as well as others, see [4, 2, 1].

A excellent, radically new, approach to ultrafilters, ultraproducts, ultraactegories, and pretoposes can be found in [5] where the author also foresees a possible comparison of the two original notions of ultracategories in future work.

In this work, we introduce a colax idempotent pseudomonad on an ***ultracompletion*** 2-functor on the 2-category **Cat** of small categories. Given a (small) category $\mathcal{C}$, write $\mathfrak{U}(\mathcal{C})$ for the category which consists of following data:

Objects are triples $(I,\mathcal{U},(c_i)_{i\in I})$ where $\mathcal{U}$ is an ultrafilter on the set I, and $(c_i)_{i\in I}$ is an I-indexed family of objects in $\mathcal{C}$.

An arrow $[V,f,(g_v)_{v\in V}]\colon(I,\mathcal{U},(c_i)_{i\in I})\to(J,\mathcal{V},(d_j)_{j\in J})$ is represented by a triple of a set $V\in\mathcal{V}$, a function $f\colon V\to I$ such that the inverse image of a set in $\mathcal{U}$ is a set in $\mathcal{V}$[1], and a family $(g_v\colon c_{f(v)}\to d_v)_{v\in V}$ of arrows in $\mathcal{C}$. Two representatives $(U,f,(g_v)_{v\in V})$ and $(U',f',(g'_v)_{v\in V'})$ are equivalent if $g_v=g'_v$ for all $v\in V\cap V'$.

Composition of arrows is given componentwise.

Remark. Let $\mathcal{T}$ denote a terminal category. The ultracompletion $\mathfrak{U}(\mathcal{T})$ is (equivalent to) the opposite of the category $\mathcal{UF}$ of ultrafilters of [5]. More generally, $\mathfrak{U}(\mathcal{C})$ is equivalent to $\left(\mathcal{UF}_{\mathrm{Fam}(\mathcal{C}^{\mathrm{op}})}\right)^{\mathrm{op}}$, where Fam is the usual coproduct completion of a category.

The assignment $\mathcal{C}\longmapsto\mathfrak{U}(\mathcal{C})$ extends to a 2-functor $\mathfrak{U}\colon\mathbf{Cat}\longrightarrow\mathbf{Cat}$, which we call ***ultracompletion***.

We briefly introduce the rest of the structure on the ultracompletion functor (write T for a fixed one-element set): for a fixed category $\mathcal{C}$, the unit functor $\nu_{\mathcal{C}}\colon\mathcal{C}\longrightarrow\mathfrak{U}(\mathcal{C})$ takes an object c to the triple $(T,\{T\},(c))$ consisting of a one-object family. The multiplication functor

$$\begin{array}{ccc}\mathfrak{U}(\mathfrak{U}(\mathcal{C})) & \xrightarrow{\qquad\mu_{\mathcal{C}}\qquad} & \mathfrak{U}(\mathcal{C})\\ (I,\mathscr{U},(J_i,\mathscr{V}_i,(c_j)_{j\in J_i})_{i\in I}) & \longmapsto & (\sum_{i\in I}J_i,\sum_{\mathscr{U}}\mathscr{V}_i,(c_{(i,j)})_{(i,j)\in\sum_{i\in I}J_i})\end{array}$$

[1]In other words, $f^{-1}\colon\wp(I)\to\wp(J)$ maps $\mathcal{U}\subseteq\wp(I)$ into $\mathcal{V}\subseteq\wp(J)$, see [5].

which employs the indexed sum of ultrafilters, see [5]. It is easy to see that they provide the data for a pseudomonad U on $\mathcal{C}at$. Finally we introduce a natural family of natural transformations

$$\mathfrak{U}(\mathcal{C}) \underset{\nu_{\mathfrak{U}(\mathcal{C})}}{\overset{\mathfrak{U}(\nu_{\mathcal{C}})}{\underset{\lambda_{\mathcal{C}}\Uparrow}{\rightrightarrows}}} \mathfrak{U}(\mathfrak{U}(\mathcal{C})) \qquad \begin{array}{l} (I,\mathcal{U},(c_i)_{i\in I}) \longmapsto (I,\mathcal{U},((T,\mathcal{T},(c_i)))_{i\in I}) \\ \qquad\qquad \uparrow [I,!,[T,\mathrm{k}_i,(\mathrm{id}_{c_i})]_{i\in I}] \\ (I,\mathcal{U},(c_i)_{i\in I}) \longmapsto (T,\mathcal{T},(I,\mathcal{U},(c_i)_{i\in I})) \end{array}$$

where $\mathrm{k}_i\colon T \to I$ is the constant function with value i.

Theorem. *The quadruple* $\mathrm{U} := (\mathfrak{U},\mu,\nu,\lambda)$ *is a colax idempotent pseudomonad on* **Cat**.

The ultracompletion functor can be connected with both notions of ultracategories. For the sake of clarity, we shall denote by M-**Ultcat**, the 2-category of ultracategories, ultrafunctors, and natural ultra-transformations in the sense of Makkai's [8], and by L-**Ultcat**, the 2-category of ultracategories, ultrafunctors, and natural ultra-transformations in the sense of Lurie's [7].

Proposition. *Let* $\mathcal{C}$ *be a category.*
(i) The category $\mathfrak{U}(\mathcal{C})$ *is an ultracategory in the sense of Makkai, and the 2-functor* $\mathfrak{U}\colon \mathbf{Cat} \longrightarrow \mathbf{Cat}$ *factors through the forgetful 2-functor* M-$\mathbf{Ultcat} \longrightarrow \mathbf{Cat}$.
(ii) The category $\mathfrak{U}(\mathcal{C})$ *is an ultracategory in the sense of Lurie, and the 2-functor* $\mathfrak{U}\colon \mathbf{Cat} \longrightarrow \mathbf{Cat}$ *factors through the forgetful 2-functor* L-$\mathbf{Ultcat} \longrightarrow \mathbf{Cat}$.

Corollary. *Each* U*-pseudoalgebra* $\mathfrak{U}(\mathcal{C}) \xrightarrow{\alpha} \mathcal{C}$ *bears a structure of ultracategory in the sense of Makkai, and a structure of ultracategory in the sense of Lurie, in such ways that each assignment extends to a faithful 2-functor from* U-**PsAlg** *into* M-**Ultcat** *and into* L-**Ultcat**, *respectively.*

Finally, we have a result along the lines of Theorem 4.1 of [8].

Theorem. *Let* $\mathcal{P}$ *be a pretopos. The evaluation functor* $\mathrm{ev}\colon \mathcal{P} \longrightarrow \mathrm{U}(\mathrm{PreTop}(\mathcal{P},\mathbf{Set}),\mathbf{Set})$ *is an equivalence of categories.*

The next steps will consider more closely the relationship between U-pseudoalgebras and ultracategories in the sense of Makkai, the connections with the work of Garner's in [5], and the abstract part of duality in line with previous work as in [3, 6, 9].

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