A categorical view on signatures for inductive types

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Abstract.
Higher inductive-inductive types form a very general class of inductive types that seems to suffice to define all higher inductive types considered in the Homotopy Type Theory book [4], including the Cauchy reals. In [1] Kaposi and Kovács describe a way to specify higher inductive-inductive types (HIIT) in type theory. Kaposi and Kovács specify HIIT's using a *theory of signatures*: a specific type theory where contexts are signatures that encode a particular HIIT. We can morally see a signature for an inductive type as the presentation of some algebraic theory, and a theory of signatures as a class of algebraic theories.

In this talk, we will present a categorical analysis of these theories of signatures. We will first analyse the prototypical example of closed finitary inductive-inductive types, which correspond to finite presentations of generalized algebraic theories without equations. Looking at the theory of signatures we recover a strict version of Uemura's *representable map category* for the universal exponentiable arrow [3]. This motivating case of closed finitary inductive-inductive can then be extended; by enriching the theory of signatures on one side we get an enhanced notion of representable map category on the other side.

For example, going from closed signatures to open signatures corresponds to a fibered notion of representable map categories. More interestingly from the homotopical point of view, extending the signature from inductive-inductive to *higher* inductive-inductive corresponds to taking ∞-representable map categories [2].

Such a theory of signatures gives a syntactic specification for inductive types, which we can then interpret in any suitable target type theory. This should allow us to compare the strength of signatures and give an analysis of decompositions of signatures into basic type formers.

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References

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