The derivator associated to a dg-category

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Abstract.

Derivators, introduced independently by Grothendieck, Heller, Franke and further developed by Groth (see [4], [3]), yield a model of higher categories based on the language of 2-categories. A *prederivator* is a 2-functor

$$\mathbb{D}\colon \mathbf{Cat}^{\mathrm{op}} \to \mathbf{CAT}, \qquad I \mapsto \mathbb{D}(I).$$

and a *derivator* is a prederivator with additional properties. Here, we denote by **Cat** the 2-category of small categories and by **CAT** the 2-category of large categories – we also disregard logic and "size" issues here.

Heuristically, a derivator can be viewed as a collection of "homotopy categories of diagrams". A typical example is the *derivator associated to an* ∞*-category* $\mathscr{C}$, defined as follows:

$$\mathbb{D}_{\mathscr{C}}(I) = h(\mathscr{C}^I), \tag{0.1}$$

where $\mathscr{C}^I$ denotes the ∞-category of functors from (the nerve of) I to $\mathscr{C}$ and $h(-)$ denotes the homotopy category. Notice that, by taking $I = e$ the terminal category, we have $\mathbb{D}(e) = h(\mathscr{C})$. The homotopy category of $\mathscr{C}$ itself does not contain enough information to reconstruct $\mathscr{C}$, but the derivator $\mathbb{D}_{\mathscr{C}}$ does, in some sense (see for instance [2]). The properties we require of derivators essentially allow us to define *homotopy Kan extensions* and in particular *homotopy limits and colimits*. This is crucial in any version of higher category theory.

Higher categories, and hence also derivators, have a wide range of applications. In particular, they naturally appear in *homotopical and homological algebra*. If A is a ring, we may investigate its properties by introducing its *derived category* $\mathsf{D}(A)$. $\mathsf{D}(A)$ is defined as the localization of the category of chain complexes of A-modules along quasi-isomorphisms, namely, morphisms inducing isomorphisms in cohomology. The derived category $\mathsf{D}(A)$ is a *triangulated category*, which means that it has an additional structure allowing us to compute some homotopy limits and colimits in the form of *mapping cones*. Unfortunately, not all homotopy limits and colimits can be computed inside a triangulated category, and – even worse – they are *not functorial*.

What is true is that triangulated categories are almost always homotopy categories of higher categories; such higher categories, called *enhancements*, have the additional properties of being *stable*, which means that they in some sense behave like abelian categories. We have a theory of *stable* ∞*-categories* and, not surprisingly, a theory of *stable derivators* [3]. If $\mathscr{C}$ is a stable ∞-category, the

derivator $\mathbb{D}_{\mathscr{C}}$ (cf. (0.1)) is indeed a stable derivator. The derived category $\mathsf{D}(A)$ has a natural stable ∞-categorical enhancement, so it has also a stable derivator enhancement.

One might be content with the above picture, but there is catch. For a triangulated category, the most natural higher categorical enhancement is not described as a stable ∞-category or a stable derivator, but as a *differential graded (dg-) category.* A dg-category is a category *enriched in chain complexes.* In particular, if $\mathscr{A}$ is a dg-category, we may take its *homotopy category* $H^0(\mathscr{A})$ just by taking zeroth cohomology of hom complexes. If A is a ring, we may easily define its *derived dg-category* $\mathsf{D}_{\mathrm{dg}}(A)$, for which the equivalence $H^0(\mathsf{D}_{\mathrm{dg}}(A)) \cong \mathsf{D}(A)$ holds. Dg-categories are in fact yet another model of higher categories, one which is best suited for applications to homological algebra.

Inside a given dg-category $\mathscr{A}$, we can define well behaved *homotopy limits and colimits*, and *functorial mapping cones.* A dg-category having such mapping cones is called *pretriangulated.* This is the differential graded version of stability: the homotopy category $H^0(\mathscr{A})$ of a pretriangulated category $\mathscr{A}$ has a natural structure of triangulated category.

Now, we know that we can define a (pre)derivator associated to an ∞-category $\mathscr{C}$. If we start from a pretriangulated dg-category $\mathscr{A}$, we may take its *dg-nerve* $\mathrm{N}_{\mathrm{dg}}(\mathscr{A})$, which is a stable ∞-category, and then the associated (pre)derivator; still, in the existing literature there is no direct construction of a *(pre)derivator associated to a dg-category.* This work will close this gap. If $\mathscr{A}$ is a dg-category, we define:

$$\mathbb{D}_{\mathscr{A}}(I) = H^0(\mathbb{R}\mathrm{Hom}(I, \mathscr{A})), \tag{0.2}$$

where $\mathbb{R}\mathrm{Hom}(I, \mathscr{A})$ denotes the dg-category of *quasi-functors* [1] between the (free linear category generated by) I and $\mathscr{A}$. Quasi-functors are essentially "homotopically coherent functors" between dg-categories.

Our main result is that the above formula (0.2) yields a stable derivator, assuming that $\mathscr{A}$ is pretriangulated and *homotopy complete and cocomplete.* To show this, we develop a new theory of *homotopy limits and colimits* in dg-categories. We will also discuss some interesting applications to [5, Theorem 4.2] and to Gorenstein projective modules. In particular, thanks to the Ph.D. thesis of H.Holm, we know that the dg-category of totally acyclic complexes of projective modules is a dg-enhancement of the stable category of Gorenstein projective modules. For a small category I and a suitable ring R with finitely many objects, we can define an algebra RI. We aim to show that the derivator associated to the dg-category of totally acyclic complexes of projective R-modules, evaluated in I, is equivalent to the category of totally acyclic complexes of projective RI-modules.

References

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