Toposes as standard universes

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Abstract.

E. Nelson introduced Internal Set Theory (IST) in 1977 [5], in an attempt to make the methods of nonstandard analysis more accessible to mathematicians (and physicists) not acquainted with Logic, particularly model theory. His approach was to extend ZFC by adding a new unary 'standardness' predicate $\mathrm{st}(x)$ and three axiom schemata (chiefly Transfer) to govern the behaviour of this new notion. This provides a reasonably contained set of rules one can use to make new proofs. The resulting theory is a conservative extension of ZFC, so that it raises no new foundational issues and can be used to prove classical results: any theorem of IST that can be stated in the language of ZFC is provable in ZFC, even if the IST proof cannot be expressed in ZFC.

There were several attempts to categorify nonstandard proof methods (e.g., [4, 7, 8]), but this talk follows a new perspective which is rather natural and addresses all three schemata from IST instead of focusing on just one in isolation. The point of view is that the additional axiom schemata of Internal Set Theory express relationships between hyperdoctrines (some of which are triposes [6]), envisioned as tools allowing us to abstract away from the ideas of internal formula, internal formula with standard parameters, and external formula while preserving their logical features. Starting from set theory as a template also allows us to leverage the well-known connections between topos theory and set theory [1, 2, 3, 9] — taking the "internal universe" to be an elementary topos with extra structure ought to be seen as a straight-up generalisation of starting from a model of ZFC with an additional predicate subject to some axioms.

Following [10], this talk will focus on toposes that closely resemble the set-theoretic models of IST by admitting a notion of 'standard element'. We will discuss the structure needed to implement transfer, standardisation, and idealisation internally to a topos, and sketch the proof that any topos satisfying the internal axiom of choice occurs as a universe of standard objects and maps [10]. This development allows one to employ these proof methods in environments such as toposes of G-sets and Boolean étendues.

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