Ordinals as Coalgebras: some missing categorical techniques

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Abstract.

Simply working out the characterisation of a standard categorical notion in a specific category often reproduces the textbook account of an old subject, or guides us in developing a new one.

Here we consider notions of "ordinal", using the theory of extensional well founded coalgebras, with the "down-sets" functor $\mathcal{D}$ on posets for the covariant powerset $\mathcal{P}$ in the original set-based example.

However, recovering the popular notion (a transitive extensional well founded relation) is not the easy application that we hoped for. It raises categorical questions that are simply stated and could be widely applicable but seem to be unknown.

Any binary relation $(\prec) \subset X \times X$ can be expressed as a coalgebra $\alpha : X \to \mathcal{P}X$. It is extensional iff α is mono and well-foundedness can be characterised by a "broken pullback" that I have discussed at categorical meetings. In our subject everything is up for negotiation: not only the category and the endofunctor but also the notion of "mono", which we replace with a factorisation system.

For the down-sets functor, a (well founded) coalgebra is a set with two binary relations $(X, \leq, \prec)$, where $(\leq)$ is a partial order, $(\prec)$ is a (well founded) binary relation and these are *compatible*,

$$z \leq y \prec x \Longrightarrow z \prec x \qquad \text{and} \qquad z \prec y \leq x \Longrightarrow z \prec x.$$

Then $f : Y \to X$ is a $\mathcal{D}$-*(coalgebra) homomorphism* iff

$$\begin{array}{llcl} \forall y y' : Y. & y' \leq_Y y & \Longrightarrow & f y' \leq_X f y \\ \forall y y' : Y. & y' \prec_Y y & \Longrightarrow & f y' \prec_X f y \\ \forall x : X. \forall y : Y. & x \prec_X f y & \Longrightarrow & \exists y' : Y. x \leq_X f y' \ \wedge \ y' \prec_Y y, \end{array}$$

whereas in the **Set** version we just had $x = f y'$, which we call a $\mathcal{P}$-*homomorphism*.

For the "monos", a categorist unencumbered by the historical baggage of set theory would use *lower sets*. This (easily) reproduces the (difficult) theory of *plump* ordinals in my 1996 JSL paper and has the universal property (*transfinite recursion*) with monotone successor that Joyal and Moerdijk identified in their contemporary *Algebraic Set Theory*. Plump ordinals grow very fast: $\omega \cdot 2$ does not exist in the simplest non-classical topos $\mathbf{Set}^{\to}$ without Replacement. You might think this situation is good or bad, but I intend to develop it into a purely categorical understanding of that logical principle.

For the popular notion we try using *regular* monos or full inclusions to redefine extensionality. Then $(\leq)$ is "set-theoretic inclusion" $(\subseteq)$ derived from $(\prec)$, which must be *meta*-transitive,

$$\forall w, x, y. \quad (\forall z. z \prec y \Rightarrow z \prec x) \ \wedge \ (x \prec w) \Longrightarrow (y \prec w).$$

The familiar *one-point successor* preserves this and there is a *rank* operation, *i.e.* a left adjoint to the inclusion amongst all well founded coalgebras. However, binary joins are very badly behaved and I don't know what transfinite recursion theorem might hold.

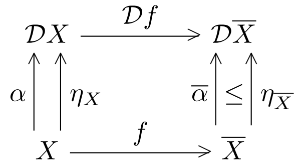
For an ordinary transitive $(\prec)$, its reflexive closure $(\preceq)$ is the appropriate choice for $(\leq)$, because then all $\mathcal{D}$-homomorphisms are actually $\mathcal{P}$-homomorphisms and lower inclusions. Binary joins (but with respect to $(\subseteq)$) are nicely behaved and transfinite recursion holds with inflationary successor.

But in developing the rank functor, we must consider (ordinary) extensionality and transitivity separately, falling back on symbolic methods and getting little benefit from known categorical theory.

We call a $\mathcal{D}$-coalgebra *transitive* if $\forall xy.\ y \prec x \Longrightarrow y \leq x$, or $\alpha \leq \eta_X$.
This fits neatly with $(\mathcal{D}, \eta, \mu)$ being a Kock–Zöbelein monad, *i.e.* with $\mathcal{D}\eta_X \leq \eta_{\mathcal{D}X}$.

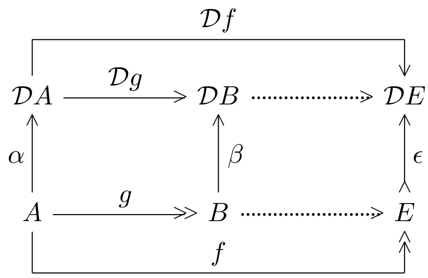
How can the transitive closure of general coalgebras be expressed 2-categorically?

It should be something like a *co-inserter*, but that's not it and, when I asked a senior 2-categorist, he didn't know what it was.



In the general theory, the extensional ("Mostowski") quotient is given by the fixed point of repeated factorisation of the structure map using the chosen notions of mono and epi. When the functor preserves the monos, this is actually just the longest corresponding epi.

However, $\mathcal{D}$ does *not* preserve plain monos, so to find the fixed point we need my notion of "well founded element". We call a regular epi homomorphism $g : A \twoheadrightarrow B$ *well projected* if it factors uniquely into every regular epi homomorphism $f : A \twoheadrightarrow E$ with E extensional:



How are well projected maps characterised order-theoretically?

So this piece of category theory has not fitted well with the traditional notion, but that could be the fault of the tradition.

In any case, constructively, the popular notion does not capture the more "combinatorial" ideas of ordinals used in subjects such as proof theory. That is because the functors $\mathcal{P}$ and $\mathcal{D}$ still use full higher order logic. But the reason for using category theory to generalise ideas is that quite different, maybe more combinatorial, functors could be used instead and would give entirely different results.

Relevant papers and seminar slides are at `www.PaulTaylor.EU/ordinals/`