Enriched universal algebra

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Abstract.

Universal algebra, introduced by Birkhoff, deals with sets A equipped with functions $f_A \colon A^n \to A$, where f is a function symbol of arity n in a certain *language* (or signature) $\mathbb{L}$. Starting from this, one builds *terms* and *equations*, and studies the classes of algebras satisfying a given set of equations.

A categorical treatment of universal algebra was given by Lawvere through the concept of *Lawvere theory*. This was further developed in the context of *enriched categories* by several authors; including Lack and Power, Bourke and Garner, and Lucyshin-Wright and Parker. Such generalizations follow the categorical approach of Lawvere, but do not yet provide a notion of *enriched universal algebra* with function symbols, recursively generated terms, and equations. In fact, instances of this have been developed only in specific situations: notably over posets, metric spaces, and complete partial orders.

In this talk, mostly based on a joint paper with Rosický, we unify this fragmented picture under the same general theory, providing new useful tools that will allow the development of universal algebra in other areas of enriched category theory.

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Our starting point is a *language* $\mathbb{L}$ which consists of a set of (X, Y)-ary function symbols, whose input and output arities are objects of the base of enrichment $\mathcal{V}$. Then we define $\mathbb{L}$-*structures* and $\mathbb{L}$-*terms*; the latter are constructed recursively out of the function symbols of $\mathbb{L}$, the morphisms of $\mathcal{V}$, and by incorporating the monoidal closed structure of $\mathcal{V}$. Interpretation of terms in $\mathbb{L}$-structures comes next:

$$t \in (X, Y)\text{-ary term},\ A \in \mathbb{L}\text{-structure} \quad \mapsto \quad t_A \colon A^X \to A^Y \text{ in } \mathcal{V}.$$

And finally, an *equational* $\mathbb{L}$-*theory* $\mathbb{E}$ is defined as a family of equations $\{s_j = t_j\}_{j \in J}$ between terms of the same arity; models of $\mathbb{E}$ are $\mathbb{L}$-structures satisfying the interpreted equations.

If the base category $\mathcal{V}$ is locally finitely presentable as a closed category, we can talk about *finitary* equational theories just by restricting the input arities to vary among the finitely presentable objects of $\mathcal{V}$. In the same vein, one can define Φ-ary equational theories for a sound class of weights Φ.

Then, our first result shows that $\mathcal{V}$-categories of models of finitary equational theories can be equivalently described as $\mathcal{V}$-categories of algebras for finitary enriched monads on $\mathcal{V}$, generalizing the ordinary results for finitary varieties. In the sound case, the $\mathcal{V}$-categories of models of Φ-ary equational theories are the $\mathcal{V}$-categories of algebras of Φ-ary enriched monads on $\mathcal{V}$.

Secondly, we determine the simplest set of output arities that are necessary to express models of equational theories. In particular we will see why in the case of $\mathcal{V} = \mathbf{Set}, \mathbf{Pos}, \mathbf{Met}$, and ω-$\mathbf{CPO}$ it is enough to consider terms with terminal output arity, and we will get hints on how to develop new applications, including for instance 2-categorical and simplicial universal algebra.