Cauchy convergence for normed categories

W. Tholen

Maria Manuel Clementino (mmc@mat.uc.pt)
Universidade de Coimbra

Dirk Hofmann (dirk@ua.pt)
Universidade de Aveiro

Walter Tholen (tholen@yorku.ca)
York University, Toronto

Abstract. Building on the notion of normed category as suggested by Lawvere [1], we introduce notions of Cauchy convergence and cocompletenes for such categories which differ from proposals in previous works, such as [2]. Key to our approach is to treat them consequentially as categories enriched in the monoidal-closed category of normed sets, *i.e.*, of sets which come with a norm function. Our notions largely lead to the anticipated outcomes when considering individual metric spaces or normed groups as small normed categories (in fact, groupoids), but they can be quite challenging when trying to establish them for large categories, such as those of semi-normed or normed vector spaces – not just because norms of vectors need to be allowed to have value ∞ in order to guarantee the existence of colimits of (sufficiently many) infinite sequences.

The interesting large normed categories typically have objects with some quantitative structure which, however, gets largely ignored by their morphisms, such as normed vector spaces with all linear maps. But the object structure is then used to declare norms of morphisms which enable one to identify meaningful types amongst them, just as the usual operator norm identifies bounded or 1-Lipschitz operators of vector spaces. Working with a general commutative quantale $\mathcal{V}$, rather than only with Lawvere's quantale $\mathcal{R}_+$ of real numbers, we will demonstrate that the categorically atypical and, in fact, questionable structure gap between objects and morphisms is already visible in the underlying normed category of the enriching category of $\mathcal{V}$-normed sets. To show that this normed category and, in fact, all presheaf categories over it, are Cauchy cocomplete, we assume the quantale $\mathcal{V}$ to satisfy alternatively a couple of light extra properties which, however, are present in all instances of interest to us. Of utmost importance to the general theory is the fact that our notion of Cauchy convergence is subsumed by the notion of weighted colimit of enriched category theory. With this theory and, in particular, with a result of [3], we are able to prove that all $\mathcal{V}$-normed categories have Cauchy cocompletions, for $\mathcal{V}$ satisfying our alternative light assumptions.

(We emphasize that our notion of Cauchy cocompleteness is not to be confused with the selfdual idempotent-split property of a category, often referred to as Cauchy completeness. Time permitting, we will comment on the connection between the two notions. For all details on this and on any other facts and examples, we must refer to the forthcoming [4].)

References

[1] F.W. Lawvere, *Metric spaces, generalized logic, and closed categories*, Rend. Sem. Mat. Fis. Milano 43, 135–166, 1973. Republished in: Reprints in Theory Appl. Categories 1, 2002.

[2] W. Kubiś, *Categories with norms*, preprint, arXiv:1705.10189v1 [math.CT], 2017.

[3] G.M. Kelly and A. Schmitt, *Notes on enriched categories with colimits of some class*, Theory Appl. Categories 14, no. 17, 399-423.

[4] M.M. Clementino, D. Hofmann and W. Tholen, *Cauchy convergence in $\mathcal{V}$-normed categories*, in preparation.