Operads colored by categories

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Abstract. Colored operads, also known as symmetric multicategories, have proven to be useful in various mathematical disciplines, such as algebraic topology, algebraic geometry, homotopy theory, or mathematical physics.

For a set C, a C-colored operad P consists of objects $P(c_1 \cdots c_n; c)$ of abstract n-ary operations, whose inputs and output have specific types $c_1, \ldots, c_n, c \in C$, together with associative and unital composition maps. We consider a generalisation, where the colors form a category; the morphisms of the coloring category act on inputs and output of the operations, possibly changing their type. Such generalisation of colored operads originally appeared in [DS03] under the name 'symmetric substitudes', and later independently in [Petersen13, Ward22]. There are applications of category-colored operads in deformation theory [DSVV24] and in homotopy theory [BW22].

I will present a new definition based on partial compositions, which is suitable for non-unital version of operads. The idea is to replace the sources of the operad composition

$$P_n \otimes P_m \xrightarrow{\circ_i} P_{n+m}$$

by certain colimits $P_n \otimes_i P_m$, which deal with the categorical coloring. This approach leads us to realize that:

Result I: *Category-colored operads are internal algebras of a certain categorical operad of functors.*

In the homotopy theory of algebras, it is standard to encode an algebra as an algebra of an operad. If this operad is binary quadratic (meaning it is free modulo quadratic relations and generated by binary operations) and 'Koszul', there is an algorithm for finding its minimal cofibrant replacement. Algebras of this replacement are then homotopy versions of the original algebras [Markl04]. To imitate this process with operads in place of algebras, it was essential to pass to the category-colored setting, which resulted in:

Result IIa: *There is a quadratic binary non-unital operad, colored by a groupoid of permutations Σ, whose algebras are non-unital symmetric operads.* This directly extends to:

Result IIb: *There is a quadratic binary non-unital category-colored operad, whose algebras are non-unital Markl $\mathbb{O}$-operads for an operadic category $\mathbb{O}$.*

Operadic categories were introduced in [BM15] as a unifying framework for various operadic structures (such as cyclic and modular operads, di–operads, &c.) and their homotopy theory.

The **Results** can be found in my recent article [T23], on which the talk is based.

References

[BM15] M.A. Batanin and M. Markl. Operadic categories and duoidal Deligne's conjecture. *Advances in Mathematics*, 285:1630–1687, 2015.

[BW22] M.A. Batanin and D. White. Homotopy theory of algebras of substitudes and their localisation. *Transactions of the American Mathematical Society*, 375(05):3569–3640, 2022.

[DS03] B. Day and R. Street. Abstract substitution in enriched categories. *Journal of Pure and Applied Algebra*, 179(1-2):49–63, 2003.

[DSVV24] V. Dotsenko, S. Shadrin, A. Vaintrob, and B. Vallette. Deformation theory of cohomological field theories, *Journal für die reine und angewandte Mathematik (Crelles Journal)*, ahead of print, 2024.

[Markl04] M. Markl. Homotopy algebras are homotopy algebras. *Forum Mathematicum*, 16:129–160, 2004.

[Petersen13] D. Petersen. The operad structure of admissible G-covers. *Algebra & Number Theory*, 7(8):1953–1975, 2013.

[T23] D. Trnka. Category-colored operads, internal operads, and Markl O-operads, *Theory and Applications of Categories*, 39:874–915, 2023.

[Ward22] B. C. Ward. Massey Products for Graph Homology, *International Mathematics Research Notices*, 2022(11):8086-8161, 2022.