Formal theory of Rezk completions

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Abstract.

Our purpose is to illuminate and to extend the theory of *univalent categories* in homotopy type theory. To this end, we generalize from the bicategory Cat and univalent categories therein to "Cat-like" 2-categories, equipped with a Yoneda structure, and univalent objects therein. In particular, we generalize and refine the result that weak equivalences (between univalent categories), are necessarily isomorphisms (adjoint equivalences). Consequently, we conclude that any fully faithful and (mere) essentially surjective functor induces an isomorphism between the univalent completions (i.e., the Rezk completion) of the source and target object.

Univalent category theory. In univalent foundations, every mathematical object comes equipped with its notion of sameness, and reasoning is invariant under this notion. The univalence axiom implies that univalent categories are necessarily invariant under weak equivalences. Therefore, the most useful notion of category is that of univalent category, in univalent foundations. However, not every *classical* construction is closed under univalence, such as Kleisli categories constructed via Kleisli morphisms. That is, the Kleisli category (as before) is not-necessarily univalent, even if the underlying category is univalent. A process to turn a non-univalent category into a univalent one is provided in [2] ; it provides a construction of the "free univalent" category, referred to as the *Rezk completion.* Concretely, the Rezk completion of a category is constructed as the full subcategory of *representable* presheaves. The universal property of the Rezk completion [2] , says precisely that unit of associated adjunction (given by the inclusion $\mathsf{Cat}_{\mathsf{univ}} \hookrightarrow \mathsf{Cat}$) is a pointwise weak equivalence. The pointwise weak equivalences are shown to correspond with the fully faithful and essentially surjective functors. (the latter are referred to as the *weak equivalences* of categories.)

Our work. In this abstract, we generalize the results concerning the theory of univalent categories, weak equivalences and Rezk/univalent completions, see [2, 5]. To achieve this generalization we build upon an existing theory of (univalent) bicategories [4] . Additionally, we formulate a type theoretic version of Yoneda structures [1] . This study requires to reason about categories modulo isomorphism, instead of equivalence. Hence, we consider those bicategories equipped with an "underlying precategory", referred to as 2-categories. We universally characterize the essentially surjective functors relative to a Yoneda structure, generalizing the proof that the weak equivalences of categories are the pointwise weak equivalences.

Yoneda structures. From now on, we fix a 2-category $\mathcal{K}$ equipped with a Yoneda structure, see [1] . (Informally, the objects of $\mathcal{K}$ are to be interpreted as (V-enriched) 1-categories.) The Yoneda structure on $\mathcal{K}$ assigns to every object X an object $\mathcal{P}\,X$ (its object of presheaves) and a morphism $よ_X : X \to \mathcal{P}\,X$ (its Yoneda morphism), see [1] for the universal property of $(X, よ_X)$.

The main idea behind a Yoneda structure is that every morphism is uniquely determined by its action on "generalized objects" and "generalized morphisms" respectively. The idea is made formal by

the following construction, due to Street and Walters.
Construction. Every precomposition functor $\mathcal{K}(f, Z)$ factors through a displayed category over the source (hom-)category, denoted $ExNat(f, Z)$:

$$\begin{array}{ccc} & & ExNat(f,Z) \\ & \overset{(f\cdot^{e}_{Z}-)}{\nearrow} & \downarrow{\scriptstyle \pi_1} \\ \mathcal{K}(Y,Z) & \xrightarrow[(f\cdot -)]{} & \mathcal{K}(X,Z) \end{array}$$

Definitions. Let X, Y, Z be objects and $f : X \to Y$ a morphism. Then f is *essentially surjective* if $(f\cdot^{e}_{Z})$ is a weak equivalence of (univalent) categories, for every univalent Z; where an object Z is *univalent* if for any $f : X \to Y$, the category $ExNat(f, Z)$ is univalent.
Theorem. Let $f : X \to Y$ be a morphism. Assume f is a *weak equivalence*, i.e., f fully faithful [1] and essentially surjective. Then, for every univalent Z, the precomposition functor $\mathcal{K}(f, Z)$ is an isomorphism of (hom-)categories. (If f satisfies the latter, f is said to be a "local equivalence".) The converse holds if for every object, a weak equivalence into a univalent object is given.
Example. The motivating type of 2-categories are those of the form $\mathcal{K} := \mathsf{Cat}_{\mathcal{V}}$, 2-categories of $\mathcal{V}$-enriched categories ($\mathcal{V}$-functors, and $\mathcal{V}$-transformations). $\mathcal{V}$ is assumed to be strong enough as a base for enrichment. Furthermore, we require closedness and completeness in order to construct the Yoneda structure on $\mathcal{V}$.

The theorem means precisely that the "local equivalences" are given by those morphisms which are fully faithful (intuitively, "inclusions" of hom-categories) and essentially surjective on (a subtype of) objects. Consequently, the univalent objects and weak equivalences are suitably characterized internal to $\mathcal{K} = \mathsf{Cat}_{\mathcal{V}}$. Furthermore, the Rezk completion and can be constructed as the (replete) (eso,ff)-image of its Yoneda morphism, i.e., as a full subcategory of its presheaf category.
Conclusion. In this project, we provide an axiomatic framework generalizing the concrete construction of [2, 5] to a more abstract (already existing) setting: 2-categories equipped with a Yoneda structure [1] . We observe that a Yoneda structure on a 2-category provides sufficient structure to suitably interpret weak equivalences as morphisms which are essentially surjective on objects and fully faithful. This interpretation is based on (a slight generalization of) Proposition 23 in [1] . The theory presented in the framework, axiomatizes the structure a 2-category needs to have, in order to suitably construct, and reason, without worrying about "univalence-requirements".

References

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