Introduction to Stratified Toposes

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Abstract.

The notion of (elementary) topos abstracts to the level of categorical algebra several aspects of the category of sets. However, it is natural to assume the existence in the category of sets of Grothendieck universes, which is not reflected in the topos axioms.

In remedy of this, various notions of universe in a topos have been introduced. [1]'s axioms are already quite close to the present approach. The axioms of [5] are stronger than [1]'s, and draw on previous work on the semantics of the calculus of constructions, *e.g.* [2, 4]. We introduce a notion of universe in a topos that somewhat strengthens [5]'s axioms.

In our notion of universe, we will ask for a full logical inclusion of toposes

$$\mathcal{E} \hookrightarrow \mathcal{F} \quad ,$$

such that $\mathcal{F}$ admits an internal category $\mathfrak{U} \in \mathbf{Cat}(\mathcal{F})$ that 'represents' $\mathcal{E}$, in the sense that

$$\mathrm{Hom}_{\mathbf{Cat}(\mathcal{F})}(I, \mathfrak{U}) \simeq (\mathcal{F}/I)_{<\mathcal{E}} \quad ,$$

pseudonaturally in $I \in \mathcal{F}$, where $(\mathcal{F}/I)_{<\mathcal{E}}$ is the full subcategory of $\mathcal{F}/I$ on the maps with '$\mathcal{E}$-small fibers.' More intuitively, we then have, in particular, that

$$\mathrm{Hom}_{\mathbf{Cat}(\mathcal{F})}(I, \mathfrak{U}) \simeq \mathcal{E}/I \quad ,$$

pseudonaturally in $I \in \mathcal{E}$.

We will also ask for a novel density condition on the logical inclusion

$$\mathcal{E} \overset{i}{\hookrightarrow} \mathcal{F} \quad ,$$

which is used to extend the induced $\mathcal{E}$-indexed logical inclusion

$$\mathcal{E}/(-) \hookrightarrow i^*\mathcal{F}/(-)$$

to the expected $\mathcal{F}$-indexed logical inclusion

$$(\mathcal{F}/(-))_{<\mathcal{E}} \hookrightarrow \mathcal{F}/(-) \quad .$$

We introduce **stratified toposes**, which are toposes that admit a hierarchy of universes in our sense.[1] Whereas, in an ordinary topos, monomorphisms are 'represented' by the subobject classifier, in a stratified topos, *all morphisms* are, moreover, 'represented' by some universe.

[1] The term 'stratified topos' recalls the notion of stratified pseudotopos of [3], though the details of our proposal are closer to [5].

Key results about toposes can be refined to yield results about stratified toposes. As proof of concept, we construct what we call the **stratified topos of coalgebras** for a stratified Cartesian comonad on a stratified topos. This construction refines that of the topos of coalgebras for a Cartesian comonad on a topos. It also solves in our setting a problem that was left open by [3] in the setting of stratified pseudotoposes, and originally solved in [5]'s setting in the dissertation of the author [6].

References

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