CT2024

Book of Abstracts

Santiago de Compostela, Spain
23rd – 29th of June, 2024

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10:30 11:00	**Gran** Conditional flatness and fiberwise localizations in semi-abelian categories
11:00 11:30	**Coffee Break**
11:30 12:00	**Bourke** Two-dimensional limit theories enhanced
12:00 12:30	**Tendas** Enriched universal algebra
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15:30 16:00	**Fernández Fariña** Universal central extensions of braided crossed modules of groups	**Markakis** Opposites and hom weak ω-categories	**Myers** Toposes as Lex-Presentable categories
16:00 16:30	**Culot** Non-additive derived functors: a chain complex approach	**Mertens** Non-cartesian internalisation and enriched quasi-categories	**Mesiti** Aspects of 2-dimensional Elementary Topos Theory
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17:00 17:30	**Egner** Galois theory and homology in quasi-abelian functor categories	**Hadzihasanovic** Pasting diagrams beyond acyclicity	**Hermans** Virtual double categories as coloured box operads
17:30 18:00	**Vienne** Associativity of cosmash products in non-associative algebras	**Martínez-Carpena** Limit-sketchable infinity categories	**Trotta** Existential completions, AC-chaotic situations and towers of toposes

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10:30 11:00	**Cockett** Drazin inverses in categories
11:00 11:30	**Coffee Break**
11:30 12:00	**Cruttwell** Connections in algebraic geometry via tangent categories
12:00 12:30	**Koudenburg** Formal Day convolution and low-dimensional monoidal fibrations
12:30 13:00	**Kock** The universal property of the decomposition space of quasisymmetric functions

13:00 15:00	**Lunch**

	Aula 2	Aula Magna	Aula 3
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15:30 16:00	**Lanfranchi** The Grothendieck construction in the context of tangent categories	**Lindenhovius** Semicartesian categories of relations	**Ramos González** Double categorical presentations of Grothendieck topoi
16:00 16:30	**Lemay** Free differential modalities	**Ching** Differential bundles in Goodwillie calculus	**Pronk** The arrows between double category sites for Grothendieck topoi
16:30 17:00	**Coffee Break**		
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Poster session (17:00–18:00)

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Forsman	Guallart	Hautekiet	Hughes
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10:30 11:00	**Corner** A higher-dimensional Eckmann–Hilton argument
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11:30 12:00	**Cheng** Configuration spaces of points and degenerate higher categories
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10:00 10:30	**North** Coalgebraic enrichment of categorical W-types		
10:30 11:00	**Taylor** Ordinals as Coalgebras: some missing categorical techniques		
11:00 11:30	**Coffee Break**		
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12:00 12:30	**Lucatelli Nunes** Doubly-infinitary distributive categories	**Doña Mateo** Pushforward monads	**Jurka** An enriched small object argument over a cofibrantly generated base
12:30 13:00	**Ranchod** Lawvere theories and symmetric operads as substitution algebras: Free constructions for abstract syntax	**Arkor** The pullback theorem for (relative) monads	**Leoncini** Homotopy colimits enriched over a general base
13:00 15:00	**Lunch**		
15:00 15:30 15:30 16:00	**Paré** The difference calculus for functors on presheaves		
16:00 16:30	**Adámek** What is a congruence?		
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09:30 10:00	**Joyal** Free bicompletion of categories and ∞-categories
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10:30 11:00	**Berger** Perfect moment categories, cocartesain comonads, and Joyal duality
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11:30 12:00	**van den Berg** Recent progress in the theory of effective Kan fibrations in simplicial sets
12:00 12:30	**Yuksel** Toposes vs Localic Groupoids: A unified treatment of covering theorems
12:30 13:00	**Sava** The derivator associated to a dg-category

13:00 15:00	**Lunch**

	Aula 2	Aula Magna	Aula 3
15:00 15:30	**López Franco** Slack Hopf monads	**Loregian** Bicategories for automata theory	**Hora** Quotient toposes of discrete dynamical systems
15:30 16:00	**Ramos Pérez** Modules over invertible 1-cocycles	**Reader** Cotraces and inner product enrichment of bicategories	**Osmond** On a (terminally connected, pro-etale) factorization system for geometric morphisms
16:00 16:30	**Ray** Categories of modules, comodules and contramodules over representations	**Di Giorgio** First-Order Bicategories: a new categorical perspective on first-order logic	**Kawase** Formalizing accessibility and duality in a virtual equipment
16:30 17:00	**Coffee Break**		
17:00 17:30	**Femić** Gray multicategories and left and right Gray skew-multicategories	**Caviglia** 2-stacks over bisites	**Clarke** The algebraic weak factorisation system of twisted coreflections and delta lenses
17:30 18:00	**Mancini** On the representability of actions of non-associative algebras	**Ko** Limits in Enhanced Simplical Categories	**Sharma** A 2-categorical model of oriented 1-cobordisms.

	Saturday
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10:30 11:00	**Awodey** Algebraic type theory
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11:30 12:00	**Lucyshyn-Wright** V-graded categories and V-W-bigraded categories: Functor categories and bifunctors over non-symmetric bases
12:00 12:30	**Hofmann** On predicate liftings and lax extensions of functors
12:30 13:00	**Tholen** Cauchy convergence for normed categories

Invited talks

A monadic approach to non-commutative Stone duality

R. Garner

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Abstract. Boolean algebras axiomatise the theory of the set-theoretic operations $\cap$, $\cup$, $(\ )^c$ and $\varnothing$ on power-set lattices; Boolean semilattices do the same thing for the operations $\cap$, $\cup$, $\setminus$ and $\varnothing$; while the *left skew Boolean algebras* of Leech [4] do the same again for the operations $\vee, \wedge, \setminus$ and $\varnothing$ on sets of partial functions $\mathrm{Pfn}(X, Y)$ defined by:

$$\begin{aligned} f \wedge g &= f|_{\mathrm{dom} f \cap \mathrm{dom} g}\,, \\ f \setminus g &= f|_{\mathrm{dom} f \setminus \mathrm{dom} g}\,, \\ \varnothing &= \text{undefined everywhere}, \end{aligned} \qquad \text{and} \qquad f \vee g(x) = \begin{cases} g(x) & \text{if } x \in \mathrm{dom}(g) \\ f(x) & \text{if } x \in \mathrm{dom}(f) \setminus \mathrm{dom}(g) \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The axioms resemble those for Boolean semilattices; the main difference is that $\vee$ and $\wedge$ are not commutative, so that left skew Boolean algebras are a non-commutative generalisation of Boolean semilattices.

As is well known, the category of Boolean algebras is contravariantly equivalent to the category of Stone spaces, i.e., totally disconnected compact Hausdorff spaces; this is Stone duality. A very mild generalisation of this shows that Boolean semilattices are equivalent to pointed Stone spaces. Much more far-reaching is the *non-commutative Stone duality* of Kudryavtseva [3], which shows that the category of left skew Boolean algebras is equivalent to the category of sheaves on pointed Stone spaces.

Kudryavtseva's result is extremely pretty, but proving it is delicate and requires a fair amount of calculation. In this talk I will describe a different approach to its establishment which, if it perhaps does not simplify the calculations much, at least serves to justify them from category-theoretic first principles. This approach reconstructs non-commutative Stone duality from an adjunction

$$\mathbf{SBA}^{\mathrm{op}} \underset{R}{\overset{L}{\leftrightarrows}} \mathbf{Poly}$$

between the category $\mathbf{SBA}$ of left skew Boolean algebras and the category $\mathbf{Poly} = \mathbf{Fam}(\mathbf{Set}^{\mathrm{op}})$ of polynomial endofunctors of $\mathbf{Set}$, induced by homming into the polynomial $\top\colon 1 \to 2$.

As with any adjunction, one can consider the induced monad T on $\mathbf{Poly}$, and the comparison functor $K\colon \mathbf{SBA}^{\mathrm{op}} \longrightarrow \mathsf{T}\text{-}\mathbf{Alg}$. The monad T turns out to be Ellerman's *ultrasheaf monad* [1], whose algebras were characterised by Kennison [2] as the category $\mathbf{ShvKH}$ of sheaves on compact Hausdorff spaces; and the adjunction $L \dashv R$ turns out to be of *descent type*, so that $K\colon \mathbf{SBA}^{\mathrm{op}} \longrightarrow \mathbf{ShvKH}$ is fully faithful. It is now simply a matter of characterising the image of K in order to reconstruct

Kudryavtseva's result. Part of the interest here is in establishing that $L \dashv R$ is indeed of descent type; one could simply calculate away, but instead we appeal to a general, and apparently new, result which provides sufficient conditions for an adjunction to be of descent type.

References

[1] Ellerman, D. P. Sheaves of structures and generalized ultraproducts. *Annals of Mathematical Logic* (1974), 163–195.

[2] Kennison, J. F. Triples and compact sheaf representation. *Journal of Pure and Applied Algebra 20* (1981), 13–38.

[3] Kudryavtseva, G. A refinement of Stone duality to skew Boolean algebras. *Algebra Universalis 67*, 4 (2012), 397–416.

[4] Leech, J. Skew Boolean algebras. *Algebra Universalis 27*, 4 (1990), 497–506.

Dagger category theory

C. Heunen

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Abstract.

A *dagger* on a category associates to every morphism $f\colon A \to B$ a morphism $f^\dagger\colon B \to A$ going in the opposite direction in such a way that $f^{\dagger\dagger} = f$. Dagger categories are useful in many areas, including operator algebra, homological algebra, Bayesian inference, reversible computing, and quantum theory. Dagger categories can of course be studied using ordinary category theory. However, in many important ways, dagger categories behave very differently than ordinary categories. The situation compares to graph theory: directed and undirected graphs share a large part of theory, but many important results also distinguish them.

My goal in this talk is to convince you that *dagger category theory* is a very interesting area of study that relies on, but differs from, ordinary category theory. (For example, it is not just formal category theory in a universe other than **Cat** or enriched category theory over a base other than **Set**.)

We start by discussing examples. Any groupoid is an example of a dagger category, but $f^\dagger$ need not be the inverse of f; think about the transpose of a matrix, for example. The point is then made by showcasing three topics.

- The theory of *monads* works best when all structure respects the dagger: the monad and adjunctions should preserve the dagger. But for a smooth theory that is not enough. The monad and its algebras should should additionally satisfy the Frobenius law. Then any monad resolves as an adjunction, with extremal solutions given by the categories of Kleisli and Frobenius-Eilenberg-Moore algebras, which again have a dagger.
- There is a notion of *limit* for dagger categories that works well: it subsumes special cases such as dagger biproducts and dagger kernels; dagger limits are unique up to unique dagger isomorphism; a wide class of dagger limits can be built from a small selection of them; dagger limits of a fixed shape can be phrased as dagger adjoints to a diagonal functor. However, dagger categories with 'too many' dagger limits degenerate, and there is a more useful notion of dagger completeness.
- An important example is the dagger category of *Hilbert spaces*, with either continuous linear maps or linear contractions. In many ways it resembles the category of vector spaces, but it is not abelian, and the difference lies precisely in the dagger. We discuss a characterisation of this category by axioms that are elementary dagger-category-theoretic in nature and do not refer to analytic notions such as complex numbers, norm, continuity, convexity, or dimension.

References

[1] C. Heunen and M. Karvonen, *Monads on dagger categories*, Theory and Applications of Categories 31(35):1016-1043, 2016.

[2] C. Heunen and J. Vicary, *Categories for quantum theory*, Oxford University Press, 2019.

[3] C. Heunen and M. Karvonen, *Limits in dagger categories*, Theory and Applications of Categories 34(18):468-513, 2019.

[4] C. Heunen and A. Kornell, *Axioms for the category of Hilbert spaces*, Proceedings of the National Academy of Sciences 119(9):e2117024119, 2022.

[5] C. Heunen, A. Kornell, and N. van der Schaaf, *Axioms for the category of Hilbert spaces and linear contractions*, Bulletin of the London Mathematical Society 56(4):1532–1549, 2024.

[6] M. Di Meglio and C. Heunen, *Dagger categories and the complex numbers*, preprint arXiv:2401.06584, 2024.

Free bicompletion of categories and of ∞-categories

A. Joyal

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Abstract.

Whitman's theory of free lattices can be extended to free bicomplete categories. I will present an extension of this theory to free bicomplete ∞-categories. The statements and proofs are very similar. The general statement depends on the notion of ∞-category co-complete with respect to a regular class of small categories introduced by Charles Rezk.

From Yoneda's additive regular spans to fibred cartesian monoidal opfibrations: a route towards a 2-dimensional cohomology of groups

S. Mantovani

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Abstract.

In his pioneering 1960 paper [7], Nobuo Yoneda introduced the notion of additive regular span in developing a formal categorical setting useful to classify n-fold extensions in a suitable additive context. As an application of his Additive Classification Theorem, he proved that functors $\mathbf{Ext}^n : \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{A}b$, defined through connected components of n-fold extensions, are additive in both variables, with $\mathcal{A}$ abelian category. This result made it possible to introduce cohomology groups in abelian categories without projectives or injectives.

When moving to a non-additive context, as the category **Gp** of groups, to interpret cocycles we need *crossed* n-fold extensions (this approach was adopted also to give an interpretation of cohomology groups in an intrinsic context, as the one of strongly semi-abelian categories in [6], see also [1]). This change carries out the necessity of breaking the symmetry in Yoneda's setting and the need of giving a fibrational interpretation of regular spans, as decribed in [2].

The main goal of this talk is to show how it is possibile to extend Yoneda's *Additive* Classification Theorem in two different directions. Indeed, we are able to provide an abstract setting which from one side allows to treat the non-additive case, such as the one in **Gp**. On the other, it makes it possible to grow up in dimension, obtaining symmetric 2-groups in place of just abelian groups.

The key notion we eventually need is that of *fibred cartesian monoidal opfibration*

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "P"] \arrow[dr, "F"'] & & \mathcal{Y} \arrow[dl, "G"] \\ & \mathcal{B} & \end{tikzcd}$$

which turns out to be a cartesian object in the 2-category $\mathsf{OpFib}(\mathsf{Fib}(\mathcal{B}))$ of internal opfibrations in the category of fibrations over a fixed category $\mathcal{B}$. After providing a characterization of such morphisms $P\colon (\mathcal{X}, F) \to (\mathcal{Y}, G)$ in $\mathsf{OpFib}(\mathsf{Fib}(\mathcal{B}))$, it turns out that, for each b in $\mathcal{B}$, the restriction

$$P_b\colon \mathcal{X}_b \to \mathcal{Y}_b$$

is a cartesian monoidal opfibration, as defined in [4].

Joint work with A. S. Cigoli and G. Metere

In particular, when for each b in $\mathcal{B}$, P_b is an opfibration fibred in groupoids and $\mathcal{Y}_b$ is an additive category, then it is possible to show that each fibre of P_b is endowed with a structure of symmetric 2-group and, as b varies in $\mathcal{B}$, "change-of-base" functors are symmetric monoidal.

Now, suppose we start with the morphism in $\mathsf{Fib}(\mathbf{Gp})$

$$\begin{tikzcd} \mathbf{XExt} \arrow[rr, "\Pi"] \arrow[dr, "\Pi_0"'] & & \mathbf{Mod} \arrow[dl, "(\_)_0"] \\ & \mathbf{Gp} & \end{tikzcd}$$

where **XExt** is the category of crossed extensions of groups

$$X\colon \qquad 0 \longrightarrow A \xrightarrow{j} G_2 \xrightarrow{\partial} G_1 \xrightarrow{p} B \longrightarrow 1 \tag{1}$$

Mod is the category of group-modules, and the functors are given by: $\Pi(X) = (B, A)$, $\Pi_0(X) = B$, $(B, A)_0 = B$.

Here we are not in the situation described above, since, for any group B the restrictions Π_B are not fibred in groupoids. But, we can move in the desired context by factorizing Π through a suitable category of fractions, as proved in [3]. This way we can apply the results above and define, for any B-module A, its symmetric 2-group of cohomology $\mathbb{H}^3(B, A)$, whose structure is described in [5].

References

[1] D. Bourn, *The tower of n-groupoids and the long cohomology sequence*, J. Pure Appl. Algebra 62 (1989) 137–183.

[2] A. S. Cigoli, S. Mantovani, G. Metere, and E. M. Vitale, *Fibered aspects of Yoneda's regular spans*, Adv. Math. 360 (2020) 106899.

[3] A. S. Cigoli, S. Mantovani, and G. Metere, *Discrete and conservative factorizations in Fib(B)*, Appl. Categ. Structures 29 (2021), 249–265.

[4] A. S. Cigoli, S. Mantovani, and G. Metere, *On pseudofunctors sending groups to 2-groups*, Mediterr. J. Math. 20 (2023), 25pp.

[5] A. S. Cigoli, S. Mantovani, and G. Metere, *The third cohomology 2-group*, Milan J. Math. 91 (2023), 315 –330.

[6] D. Rodelo, *Directions for the Long Exact Cohomology Sequence in Moore Categories*, Appl. Categ. Structures 17 (2009), 387–418.

[7] N. Yoneda, *On Ext and exact sequences*, J. Fac. Sci. Univ. Tokyo Sect. I 8 (1960), 507–576.

The difference calculus for functors on presheaves

R. Paré

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Abstract.
We start with the one-variable case, developing a difference calculus for endofunctors F of **Set**. Like in the classical difference calculus for functions $\mathbb{R} \to \mathbb{R}$, a discrete version of the derivative, we define $\Delta[F]\colon \mathbf{Set} \to \mathbf{Set}$ by

$$\Delta[F](X) = F(X+1) \setminus F(X).$$

The "\" is set difference so $\Delta[F]$ shouldn't be expected to be functorial, but it is for a rather large class of functors, the taut functors of Manes. (A functor is taut if it preserves inverse images or, put differently, preserves pullbacks along monos.)

We develop the difference calculus for these, obtaining limit and colimit rules analogous to the classical product and sum rules. We get a lax chain rule where none exists for mere functions, and a Newton summation formula which appears as a left adjoint. Many interesting classes of functors are taut, polynomial and analytic ones for example, and for these we give explicit descriptions of their differences.

We then proceed to the multivariable case, i.e. functors between presheaf categories. The generalization of tautness we need is the preservation of complemented subobjects and their inverse images. We then get the partial differences $\Delta_A[F]$ by replacing the 1 in the definition of $\Delta[F]$ by the representable at A. All the Δ_A together form a profunctor, $\nabla[F]$, the Jacobian of F. We then establish similar rules as in the one-variable case and study some examples.

Limits and colimits in (∞, n)-category theory

M. Rovelli

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Abstract. The notion of limit for a diagram valued in an ordinary category is a very fundamental one and allows one to encode the universal property of a variety of constructions of interest. This viewpoint becomes even more relevant in the context of higher category theory, where explicit constructions are often challenging to describe, given the infinite amount of coherence involved. For this reason, it becomes crucial to have at one's disposal a meaningful and well-behaved notion for the (possibly weighted) limit of a diagram valued in an n-category or an (∞, n)-category. We will first discuss the universal property for (∞, n)-limits that naturally arises from enriched category theory (in the sense of Borceux-Kelly and Shulman), and mention some of the difficulties which arise in this particular context. We will then propose an alternative formulation, and phrase the universal property for the limit of a diagram valued in an (∞, n)-category in terms of an appropriate double $(\infty, n-1)$-category of cones over such diagram, generalizing at once the viewpoint taken for 2-limits by Grandis-Paré, for n-limits by Moser-Sarazola-Verdugo, and for $(\infty, 1)$-limits by Joyal and Lurie. This is joint work with Moser and Rasekh.

References

[1] L. Moser, N. Rasekh, M. Rovelli, (∞, n)-Limits I: definition and first consistency results, preprint arXiv:2312.11101, 2023.

Contributed talks

What is a Congruence?

J. Adámek

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Abstract.

1 Introduction

Lawvere's thesis contains a characterization of varieties of classical (finitary, one-sorted) algebras as categories with (1) coequalizers and kernel pairs, (2) an abstractly finite, regularly projective strong generator, and (3) effective congruences: every congruence (a reflexive, symmetric and transitive relation) is the kernel pair of some morphism. The concept of a congruence was generalized to the enriched setting in the fundamental paper [4], see also [5]. In case of enrichement over **Pos** (posets and monotone maps) or **Met** (metric spaces and non-expanding maps) we now present a much simpler concept called subcongruence for **Pos** and procongruence for **Met** (in order to distinguish them from the congruence in op. cit.).

2 Congruences in an order-enriched category

The role of a kernel pair of a morphism $f\colon X \to Y$ is played here by a *subkernel pair*: a universal pair $r, r'\colon R \to X$ with respect to $f \cdot r \leq f \cdot r'$. Every subkernel pair is a *relation on* X, i.e. the derived morphism $R \to X^2$ is an order-emedding, which is reflexive and transitive (and of course *not* symmetric). It is even *order-reflexive*: every parallel pair $u, u'\colon U \to X$ with $u \leq u'$ factorizes through r, r'.

A*subregular epimorphism* is a morphism which is the coinerter of a reflexive parallel pair.

Definition. A *subcongruence* on an object is an order-reflexive and transitive relation.

Let Σ be a classical (finitary) signature and denote by Σ-**Pos** the category of ordered algebras with monotone operations (and monotone homomorphisms). It has *effective subcongruences*: every subcongruence is the subkernel pair of some morphism.

More generally, every *variety of ordered algebras*, a full subcategory presented by a set of inequalities between terms, has effective subcongruences.

Here is the main result of [1] (improving that of [3]). An object is *subregularly projective* if its hom-functor preserves subregular epimorphisms.

Theorem. A **Pos**-enriched category is equivalent to a variety of ordered algebras iff it has (1) reflexive coequalizers and subkernel pairs, (2) an abstractly finite, subregularly projective strong generator, and (3) effective subcongruences.

3 Congruences in a metric-enriched category

For a real number $\varepsilon \geq 0$ the *ε-kernel pair* of a morphism $f\colon X \to Y$ is a universal pair $r_\varepsilon, r'_\varepsilon\colon R_\varepsilon \to X$ with respect to $d(f \cdot r_\varepsilon, f \cdot r'_\varepsilon) \leq \varepsilon$. Every such pair is a *relation*: the derived morphism to X^2 is an isometric embedding.

We introduce a weight $B\colon \mathcal{B}^{op} \to \mathbf{Met}$ such that every morphism f has a *kernel diagram* $D_f\colon \mathcal{B} \to \mathbf{Met}$ collecting all ε-kernel pairs of f. Every colimit of a diagram weighted by B is determined by a morphism. A *proregular epimorphism* is a morphism determining some colimit weighted by B. An object is *proregularly projective* if its hom-functor preserves proregular epimorphisms.

The kernel diagram D_f consists of parallel pairs that are reflexive and symmetric relations. They are also collectively transitive, and satisfy a continuity condition (expressing the fact that the map assigning R_ε to each $\varepsilon \geq 0$ preserves limits). A *procongruence* is a diagram weighted by B having all of those properties.

Mardare et al. [6] introduced varieties (aka 1-basic varieties) of quantitative algebras: they are categories of metric-enriched algebras presented by ε-equations between terms. We prove in [2] that up to equivalence they are precisely the **Met**-enriched categories which have (1) reflexive coequalizers and ε-coinserters, (2) an abstractly finite, proregularly projective strong generator, and (3) effective procongruences: every procongruence is the kernel diagram of some morphism.

References

[1] J. Adámek, *Categories which are varieties of classical or ordered algebras*, preprint arXiv:2402.145.

[2] J. Adámek, *Which categories are varieties of quantitative algebras?*, preprint arXiv:2402.14662v1.

[3] J. Adámek and J. Rosický, *Varieties of ordered algebras as categories*, Alg. Universalis no. 84 (2023), Paper 9.

[4] J. Bourke and R. Garner *Two-dimensional regularity and exactness*, J. Pure and Appl. Algebra no. 218 (2014), 1346–1371.

[5] A. Kurz and J. Velebil, *Quasivarieties and varieties of ordered algebras: regularity and exactness*, Mathem. Struct. Comput. Sci. no. 27 (2016), 1–42.

[6] R. Mardare, P. Panaganden and G. D. Plotkin, *On the axiomatizability of quantitative algebras*, Proceeding of Logic in Computer Science (LICS 2017), IEEE Computer Science 2017, 1–12.

Eilenberg-Moore categories and quiver representations of monads and comonads

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Abstract. Let Z be a scheme. Then, a famous result of Gabber (see, for instance, [4, Tag 077P]) shows that the category $QCoh(Z)$ of quasi-coherent sheaves over Z is a Grothendieck category. If S is a scheme and $\mathcal{Z}$ is an algebraic stack over S, the category $QCoh(\mathcal{Z})$ of quasi-coherent sheaves over $\mathcal{Z}$ is also a Grothendieck category (see, for instance, [4, Tag 06WU]).

In this paper, we prove a Gabber type result for representations of quivers in Eilenberg-Moore categories of monads. We develop a categorical framework for studying module representations taking values in Eilenberg-Moore categories of monads. For this, we generalize the usual setup of sheaves in several different ways. First, we replace the system of affine open subsets of a scheme by a quiver $\mathbb{Q} = (\mathbb{V}, \mathbb{E})$, i.e., a directed graph $\mathbb{Q}$ with a set of vertices $\mathbb{V}$ and a set of edges $\mathbb{E}$. This is motivated by Estrada and Virili [3] who studied modules over a representation $\mathcal{A} : \mathcal{X} \longrightarrow \mathit{Add}$ of a small category $\mathcal{X}$ taking values in small preadditive categories. Thereafter, we replace rings by monads over a given Grothendieck category $\mathcal{C}$. As such, we consider a representation $\mathscr{U} : \mathbb{Q} \longrightarrow Mnd(\mathcal{C})$ of the quiver $\mathbb{Q}$ taking values in the category $Mnd(\mathcal{C})$ of monads over $\mathcal{C}$. Finally, we replace the usual module categories over rings by Eilenberg-Moore categories of the monads over $\mathcal{C}$. By using systems of adjoint functors between Eilenberg-Moore categories, we obtain a categorical framework of modules over monad quivers. Our main objective is to give conditions for the category of modules over monad quivers to be Grothendieck categories, which play the role of noncommutative spaces.

We refer to a representation $\mathscr{U} : \mathbb{Q} \longrightarrow Mnd(\mathcal{C})$ as a monad quiver. To study modules over $\mathscr{U}$, we combine techniques on monads and adapt methods from earlier work in [1], [2] which are inspired by the cardinality arguments of Estrada and Virili [3]. One of our key steps is finding a modulus like bound for an endofunctor $U : \mathcal{C} \longrightarrow \mathcal{C}$ in terms of $\kappa(G)$, where G is a generator for $\mathcal{C}$ and $\kappa(G)$ is a cardinal such that G is $\kappa(G)$-presentable. As with usual ringed spaces, we have to study two kinds of

module categories over a monad quiver. The first behaves like a sheaf of modules over a ringed space. The second consists of modules that are cartesian, which resemble quasi-coherent sheaves. Another feature of our paper is that we study modules over a monad quiver in two different orientations, which we refer to as "cis-modules" and "trans-modules." We establish similar results for comodules over a comonad quiver $\mathscr{V} : \mathbb{Q} \longrightarrow Cmd(\mathcal{C})$ taking values in comonads over $\mathcal{C}$. We conclude with rational pairings of a monad quiver with a comonad quiver, which relate comodules over a comonad quiver to coreflective subcategories of modules over monad quivers.

References

[1] Banerjee, A., *Entwined modules over representations of categories*, Algebr. Represent. Theory 26 (2023), no. 6, 3185–3221.

[2] Balodi, M., Banerjee, A., and Ray, S., *Categories of modules, comodules and contramodules over representations*,Forum Math. 36,(2024), no. 1, 111–152.

[3] Estrada, S. and Virili, S., *Cartesian modules over representations of small categories*, Adv. Math. (2017), vol. 310, 557-609.

[4] The Stacks project, *Available online*, `https://stacks.math.columbia.edu/`,

The pullback theorem for (relative) monads

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Abstract.

The pullback theorem for monads states that, given a monad T on a category E, the category of T-algebras may be constructed by a certain pullback involving the category of free T-algebras, as given below [1].

$$\begin{tikzcd}
\mathbf{Alg}(T) \arrow[r, hook] \arrow[d, "u_T"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathbf{Kl}(T)^{\mathrm{op}}, \mathbf{Set}]} \arrow[d, "{[k_T^{\mathrm{op}}, \mathbf{Set}]}"] \\
E \arrow[r, hook, "y_E"'] & {[E^{\mathrm{op}}, \mathbf{Set}]}
\end{tikzcd}$$

Here, $u_T \colon \mathbf{Alg}(T) \to E$ is the forgetful functor, $y_E \colon E \to [E^{\mathrm{op}}, \mathbf{Set}]$ is the Yoneda embedding, and $k_T \colon E \to \mathbf{Kl}(T)$ is the Kleisli inclusion.

As has been repeatedly demonstrated in recent years, many classes of well-behaved monads admit a refinement of the pullback theorem, in which the category of algebras is expressed as a pullback over the nerve $n_j \colon E \to [A^{\mathrm{op}}, \mathbf{Set}]$ of some dense functor $j \colon A \to E$. (In particular, the classical pullback theorem is recovered by taking $j = 1_E$.) Such characterisations are frequently referred to as *nerve theorems*, and endow the monad and its category of algebras with particularly nice properties. Examples of classes of monads satisfying respective nerve theorems include *familially representable monads* [2], *monads with arities* [3, 4, 5], *$\mathscr{J}$-ary monads* [6], and *nervous monads* [7, 8]. It is natural to wonder to what extent these phenomena are related.

In this talk, I will explain how, by generalising the pullback theorem from monads to *relative monads* [9, 10], we may view each of the aforementioned nerve theorems as particular instances of a more (and, indeed, maximally) general phenomenon.

This talk is based on the recent preprint [11].

References

[1] F. E. J. Linton, *An outline of functorial semantics*, Seminar on triples and categorical homology theory, Springer, 1969.

[2] T. Leinster, *Nerves of algebras*, Talk presented at Category Theory 2004.

[3] M. Weber, *Familial 2-functors and parametric right adjoints*, Theory and Applications of Categories, 2007.

[4] P.-A. Melliès, *Segal condition meets computational effects*, 25th Annual IEEE Symposium on Logic in Computer Science, 2010.

[5] C. Berger, P.-A. Melliès, and M. Weber, *Monads with arities and their associated theories*, Journal of Pure and Applied Algebra, 2012.

[6] R. Lucyshyn-Wright, *Enriched algebraic theories and monads for a system of arities*, Theory and Applications of Categories, 2016.

[7] J. Bourke, and R. Garner, *Monads and theories*, Advances in Mathematics, 2019.

[8] R. Lucyshyn-Wright, and J. Parker, *Enriched structure-semantics adjunctions and monad-theory equivalences for subcategories of arities*, arXiv preprint.

[9] T. Altenkirch, J. Chapman, and T. Uustalu, *Monads need not be endofunctors*, International Conference on Foundations of Software Science and Computational Structures, 2010.

[10] N. Arkor, and D. McDermott, *The formal theory of relative monads*, Journal of Pure and Applied Algebra, 2024.

[11] N. Arkor, and D. McDermott, *The pullback theorem for relative monads*, arXiv preprint.

Algebraic Type Theory

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Abstract.

A type theoretic universe $\pi : E \to U$ in a locally cartesian closed category $\mathcal{C}$ (as in [1]) can be shown to bear an algebraic structure resulting from the type-forming operations of unit type, dependent sum, and dependent product (as shown in [2]). Specifically, the associated polynomial endofunctor $P_\pi : \mathcal{C} \to \mathcal{C}$ has the structure of a monad, for which π is itself an algebra.

This structure is here abstracted to form the concept of a "Martin-Löf algebra". Any ML-algebra is shown to model Martin-Löf type theory, and the free ones then have special type-theoretic properties. The general theory of ML-algebras is a "proof-relevant" or *categorified* version of the theory of Zermelo-Fraenkel algebras from the algebraic set theory of Joyal & Moerdijk [3].

For example, any representable natural transformation $\pi : E \to U$ of presheaves, as in [2], is necessarily *tiny* in the sense of Lawvere: the right adjoint push-forward functor $\pi_* : \mathcal{C}/E \to \mathcal{C}/U$ has a further right adjoint. It follows that the polynomial endofunctor $P_\pi : \mathcal{C} \to \mathcal{C}$ is cocontinuous and therefore admits an algebraically free monad structure, by a familiar iteration [4]. The (type theory modeled by the) colimit $\pi^\omega : E^\omega \to U^\omega$ is then the free completion under Σ-types of (that modeled by) $\pi : E \to U$. Various other type-theoretic constructions are similarly related to functorial algebraic ones.

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References

[1] S. Awodey. On Hofmann-Streicher universes, preprint arXiv:2205.10917, 2022.

[2] S. Awodey. *Natural models of homotopy type theory.* Math. Stru. Comp. Sci. 28 (2008), no. 2, pp. 241–286.

[3] A. Joyal and I. Moerdijk, *Algebraic Set Theory*, Cambridge University Press, 1995.

[4] Kelly G.M., *A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on.* Bull. Austr. Math. Soc. 22 (1980) no. 1, pp. 1–83.

Formal category theory - 50 years after

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Abstract.
Ever since the introduction of formal category theory 50 years ago by Gray in his monograph [9] there are still surprisingly few papers on the subject. I will describe the state of affairs of the subject today and compare the existing approaches to formal category theory by means of enhanced category theory which was initiated by Lack and Shulman in [12] who developed a theory of what we could call 1-enhanced 2-categories. A deeper reflection tells us that the *enhancement* itself can be considered either as

- a *property* of 1-cells in a 2-category
- a process of *assigning* properties to objects of the 2-category $\mathscr{C}at$

I will show that besides its cartesian monoidal structure, the category $\mathscr{F}$ whose objects are injective on objects and fully faithful functors which Lack and Shulman called full embeddings has a much richer structure given by other closed (but not monoidal) structures. One of the consequences of this fact is that the following notions are essentially equivalent:

(i) A 2-category with a right ideal of 1-cells

(ii) A category enriched over the closed category $\mathscr{F}_c$ whose objects are functors that are fully faithful and injective on objects which I christened - enhanced categories

In this way any 2-category with Yoneda structure can be naturally seen as an enriched category in the sense of Eilenberg and Kelly [6] and the theory developed by Street and Walters in [13] has a natural interpretation in this context. I will describe the construction of categories enriched in closed 2-categories which is complementary to the construction of Garner and Shulman [8] of categories enriched in a monoidal bicategory. Along the way, I will show how (weak) equipments in the sense of Wood [14] can also be seen as enriched categories and I will prove that they are objects of the strict double category which has a natural Gray-like tensor product. Finally, I will show how the development of these ideas give a natural solution to more than 60 years old unavailable construction by Bénabou - the reflection of morphisms of bicategories into strict ones.

References

[1] M. Bunge, J. Funk, On a bicomma object condition for KZ-doctrines, Journal of Pure and Applied Algebra 143 (1999), 69-105.

[2] D. Chikladze, Lax formal theory of monads, monoidal approach to bicategorical structures and generalized operads,Theory and Applications of Categories, Vol. 30, No. 10, (2015), 332-386.

[3] J.R.B. Cockett, J. Koslowski, R.A.G. Seely, R.J. Wood, Modules, Theory and Applications of Categories, Vol. 11, No. 17 (2003), 375-396.

[4] G.S.H. Cruttwell, M. Shulman, A unified framework for generalized multicategories, Theory and Applications of Categories 24, No. 21 (2010), 580-655.

[5] I. Di Liberti, F. Loregian, On the unicity of formal category theories, (2019), arXiv:1901.01594.

[6] S.Eilenberg,G.M. Kelly, Closed Categories, in: Eilenberg, Harrison,, MacLane, Röhrl, (eds.), Proceedings of the Conference on Categorical Algebra, (1966), 421-562.

[7] M. Fiore, N. Gambino, M. Hyland, G. Winskel, Relative pseudomonads, Kleisli bicategories, and substitution monoidal structures, Selecta Mathematica New Series 24 (2018), 2791-2830.

[8] R. Garner, M. Shulman, Enriched categories as a free cocompletion, Advances in Mathematics 289 (2016), 1-94.

[9] J.W. Gray, Formal Category Theory: Adjointness for 2-Categories, Lecture Notes in Mathematics 391, 1974.

[10] C. Hermida, Representable multicategorires, Advances in Mathematics 151 (2000), 164-225.

[11] S. R. Koudenburg, Augmented virtual double categories, Theory and Applications of Categories 35, No. 10 (2020), 261-325.

[12] S. Lack, M. Shulman, Enhanced 2-categories and limits for lax morphisms, Advances in Mathematics 229 (2012), 294-356.

[13] R. Street, R. Walters, Yoneda structures on 2-categories, Journal of Algebra 50, Issue 2 (1978), 350-379.

[14] R. J. Wood. Abstract pro arrows I, Cahiers de Topologie et Géometrie Différentielle 23 (1982), 279-290

The language of a model category

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Abstract.
Quillen model categories are a cornerstone for modern homotopy theory. These categories, originally devised to capture homotopical properties of categories like topological spaces, simplicial sets or chain complexes, have gained relevance for giving a way to construe higher categories which are of great importance, for example, in algebraic topology and geometry.

In this talk, we will see that model categories also have logical information on their own in the following sense: Given any model category, we can associate to it a class of first-order formulas referring to the fibrant objects of the category. For example, the associated language of the category of small categories, equipped with its canonical model structure, coincides with language for categories defined by Blanc [1] and Freyd [2], whose central feature is that it respects the equivalence principle.

Similarly, the language we associate to a model category respects the appropriate version of the equivalence principle: two homotopically equivalent objects satisfy the same formulas and replacing parameters by homotopically equivalent ones does not change the validity of a formula.

Finally, we will show that for M and N two Quillen equivalent model categories, their associated languages are, suitably, equivalent.

References

[1] G. Blanc, *Equivalence naturelle et formules logiques en théorie des catégories*, Archiv für mathematische Logik und Grundlagenforschung 19 (1978), no. 1, 131–137.

[2] P. Freyd, *Properties invariant within equivalence types of categories*, Algebra, Topology and Category Theory, Academic Press (1976), 55–61.

Categorical approaches to projective Fraïssé theory

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Abstract.

Fraïssé theory, understood as the study of ultrahomogeneous structures, is classical in model theory [5]. Recall that a first-order structure U is *ultrahomogeneous* if every isomorphism $f\colon A \to B$ between finitely generated substructures $A, B \subseteq U$ can be extended to an automorphism $\tilde{f}\colon U \to U$. Most classical countable ultrahomogeneous structures include the *linear order of rationals*, the *random graph*, and the *rational Urysohn metric space*.

It is natural to formulate Fraïssé theory in the language of category theory. This allows for clear and general definitions and proofs capturing the essence of the constructions involved. For example, given a pair of categories $\mathcal{K} \subseteq \mathcal{L}$, we call an $\mathcal{L}$-object U *homogeneous* in $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every pair of $\mathcal{L}$-maps from a $\mathcal{K}$-object $f, g\colon x \to U$ there is an automorphism $h\colon U \to U$ such that $h \circ g = f$. Based on work of Droste and Göbel [4] and Kubiś [7], the core of Fraïssé theory can be summarized in the following two theorems. We say that $\langle \mathcal{K}, \mathcal{L} \rangle$ is a *free completion* (or more precisely, a free sequential cocompletion) if $\mathcal{L}$ essentially arises from $\mathcal{K}$ by freely adding colimits of $\mathcal{K}$-sequences.

Theorem (Characterization of the Fraïssé limit)**.** Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a free completion and let U be an $\mathcal{L}$-object. Then the following are equivalent.

(1) U is cofinal and homogeneous in $\langle \mathcal{K}, \mathcal{L} \rangle$,

(2) U is cofinal and injective in $\langle \mathcal{K}, \mathcal{L} \rangle$,

(3) U is the $\mathcal{L}$-colimit of a Fraïssé sequence in $\mathcal{K}$.

Moreover, such U is unique and cofinal in $\mathcal{L}$, and every $\mathcal{K}$-sequence with $\mathcal{L}$-colimit U is Fraïssé in $\mathcal{K}$. Such U is called the *Fraïssé limit* of $\mathcal{K}$ in $\mathcal{L}$.

Theorem (Existence of a Fraïssé sequence)**.** Let $\mathcal{K} \neq \emptyset$ be a category. Then $\mathcal{K}$ has a Fraïssé sequence if and only if

(1) $\mathcal{K}$ is directed,

(2) $\mathcal{K}$ has the amalgamation property,

(3) $\mathcal{K}$ has a countable dominating subcategory.

Such $\mathcal{K}$ is called a *Fraïssé category.*

In 2006, Irwin and Solecki [6] introduced *projective Fraïssé theory*, where instead of embeddings of first-order structures, quotients of topological graphs are considered. The (projectively) homogeneous structure is obtained as a limit of an inverse sequence of quotient maps, instead of taking the union of

an increasing chain. The particular limit obtained by Irwin and Solecki was the Cantor space endowed with a special closed equivalence relation with the quotient space being the *pseudo-arc*, a well-known continuum. Projective Fraïssé theory fits the categorical framework presented above (with taking opposite categories everywhere), but the Fraïssé limit obtained is a pre-space (profinite space with a closed equivalence relation), not the actual space (the induced quotient).

Recently, we have considered alternative approaches to projective Fraïssé theory. With W. Kubiś we have developed Fraïssé theory of *MU-categories* [3], a specific generalization of metric-enriched categories tailored to work with categories of metrizable compacta and continuous surjections. Here the key notions are approximate in the sense that defining diagrams are not commuting exactly, but up to ε for arbitrary $\varepsilon > 0$. In this setup we have obtained the pseudo-arc as well as P-adic pseudo-solenoids as Fraïssé limits directly. Here the category of small objects $\mathcal{K}$ consists of all continuous surjections of the unit interval and all continous surjections of the unit circle whose degree uses primes only from P, respectively, so the small objects are not finite and discrete any more.

Another approach, which is a joint work in progress with T. Bice and A. Vignati [1, 2], we start with a category $\mathcal{K}$ of finite graphs, as in the classical projective Fraïssé theory, but we allow relations instead of functions as morphisms, and instead of taking the limit of an inverse sequence, we turn the sequence into a *graded ω-poset* and take its *spectrum*. This way we can construct compact spaces directly from finite graphs, but categorical limit is replaced by an ad hoc construction. Recently we found out that the spectrum can be to some extent viewed as a limit, by utilizing the notion of *lax-adjoint limits* [8] in poset-enriched categories.

In the talk I will give an overview of the three approaches and discuss new ideas about viewing the spectrum as a *unital* lax-adjoint limit.

References

[1] A. Bartoš, T. Bice, A. Vignati, *Constructing compacta from posets*, preprint arXiv:2307.01143, 2023.

[2] A. Bartoš, T. Bice, A. Vignati, *Generic compacta from relations between finite graphs*, in preparation.

[3] A. Bartoš, W. Kubiś, *Hereditarily indecomposable continua as generic mathematical structures*, preprint arXiv:2208.06886, 2022.

[4] M. Droste, R. Göbel, *Universal domains and the amalgamation property*, Math. Structures Comput. Sci. 3 (1993), no. 2, 137–159.

[5] W. Hodges, *Model theory*, Encyclopedia of Mathematics and its Applications 42, Cambridge University Press, Cambridge, 1993.

[6] T. Irwin, S. Solecki, *Projective Fraïssé limits and the pseudo-arc*, Trans. Amer. Math. Soc. 358 (2006), no. 7, 3077–3096.

[7] W. Kubiś, *Fraïssé sequences: category-theoretic approach to universal homogeneous structures*, Ann. Pure Appl. Logic 165 (2014), no. 11, 1755–1811.

[8] S. Milius, *On colimits in categories of relations*, Appl. Categ. Structures 11 (2003), no. 3, 287–312.

Invertible cells in weak ω-categories

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Abstract.
Coinductively invertible cells play a key role in the homotopy theory of strict ω-categories [3], allowing one to define weak equivalences among them. The definition of coinductively invertible cell generalises to weak ω-categories, suggesting the existence of a similar homotopy theory for weak ω-categories. Such cells have been studied by Fujii et al. [2] for Batanin and Leinster's weak ω-categories [4]. Using the new description of weak ω-categories and their computads given by Dean et al. [1], we extend and give alternative proofs of their results. We provide sufficient conditions for a cell in a weak ω-category X to be invertible, and show that those conditions are also necessary when X is free on a finite-dimensional computad. We show in particular that coherence cells and composites of invertible cells are invertible by explicitly constructing an inverse.

References

[1] C. Dean, E. Finster, I. Markakis, D. Reutter and J. Vicary, Computads for weak ω-categories as an inductive type, preprint arXiv:2208.08719, 2022.

[2] S. Fujii, K. Hoshino and Y. Maehara, Weakly invertible cells in a weak ω-category, preprint arXiv:2303.14907, 2023.

[3] Y. Lafont, F. Métayer and K. Worytkiewicz, *A folk model structure on omega-cat*, Advances in Mathematics, 2010, 224(3):1183-1231. doi:10.1016/j.aim.2010.01.007.

[4] T. Leinster, *Higher Operads, Higher Categories*, Cambridge University Press, 2004.

Perfect moment categories, cocartesian comonads and Joyal duality

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Abstract.
In [2] moment categories were introduced as small categories equipped with an *active/inert factorisation system* subject to two simple axioms. The most prominent examples are Segal's category Γ, Eilenberg's category Δ and Joyal's categories Θ_n. In this talk I will discuss *perfect moment categories* defined by the additional property that the inclusion of the active subcategory admits a left adjoint reflection such that the unit of the adjunction is pointwise inert. This notion relates on one side to a special class of cocartesian comonads, and on the other side to Barwick's perfect operator categories [1]. The latter relationship is actually a *categorical duality* which subsumes as special case Joyal's duality [3] between the n-cellular category Θ_n and the category $\mathbb{D}_n$ of combinatorial n-disks.

References

[1] C. Barwick, *From operator categories to higher operads*, Geom. & Topol. 22 (2018), 1893–1959.

[2] C. Berger, *Moment categories and operads*, Theory Appl. Categ. 38 (2022), 1485–1537.

[3] A. Joyal, *Disks, duality and θ-categories*, preprint (1997).

Two-dimensional limit theories enhanced

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Abstract.
Limit theories capture (many-sorted) sets with structure, including examples such as monoids, rings and small categories. Following Kelly [2], we can consider enriched limit theories for any suitable base of enrichment, so that when we enrich over small categories we obtain a notion of limit 2-theory, which ought to capture (many-sorted) categories with structure, such as monoidal categories, duoidal categories and double categories.

However, on a closer analysis, limit 2-theories do not not work as well as one might hope. The problems most clearly emerge in many-sorted examples of current interest such as monoidal fibrations, double categories, double fibrations and monoidal double categories where certain structural maps should be strict (and/or strictly preserved).

In this talk I will explain the problems with limit 2-theories and how they are overcome by passing to enhanced limit 2-theories, which are enriched limit theories over a different base of enrichment F [3]. Our main application is a theorem about bimodels in this setting and will explain the various equivalent descriptions of structures such as monoidal double categories [4] and double fibrations [1] in the literature.

References

[1] Cruttwell, G., Lambert, M., Pronk, D. and Szyld, M. Double fibrations. *Theory and Applications of Categories 38*, 35 (2022), 1326–1394.

[2] Kelly, G. M. Structures defined by finite limits in the enriched context. I. *Cahiers Topologie Géom. Différentielle 23*,1 (1982) 3–42.

[3] Lack, S. and Shulman, M. Enhanced 2-categories and limits for lax morphisms. *Advances in Mathematics 229*, 1 (2012), 294–356.

[4] Shulman, M. Constructing symmetric monoidal bicategories. Arxiv Preprint, 2010, arXiv:1004.0993

2-stacks over bisites

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Abstract.
Stacks generalize one dimension higher the fundamental concept of sheaf. They are pseudofunctors that are able to glue together weakly compatible local data into global data. Stacks are a very important concept in geometry, due to their ability to take into account automorphisms of objects. While many classification problems do not have a moduli space as solution because of the presence of automorphisms, it is often nonetheless possible to construct a moduli stack.

In recent years, the research community has begun generalizing the notion of stack one dimension higher. Lurie studied a notion of $(\infty, 1)$-stack, that yields a notion of $(2, 1)$-stack for a trihomomorphism that takes values in $(2, 1)$-categories, when truncated to dimension 3. And Campbell introduced a notion of 2-stack that involves a trihomomorphism from a one-dimensional category into the tricategory of bicategories.

In this talk, we will introduce a notion of 2-stack that is suitable for a trihomomorphism from a 2-category endowed with a bitopology into the tricategory of bicategories. The notion of bitopology that we consider is the one introduced by Street in [4] for bicategories. We achieve our definition of 2-stack by generalizing a characterization of stack due to Street [4].

Since our definition of 2-stack is quite abstract, we will also present a useful characterization in terms of explicit gluing conditions that can be checked more easily in practice. These explicit conditions generalize to one dimension higher the usual stacky gluing conditions. A key idea behind our characterization is to use the tricategorical Yoneda Lemma to translate the biequivalences required by the definition of 2-stack into effectiveness conditions of appropriate data of descent. As a biequivalence is equivalently a pseudofunctor which is surjective on equivalence classes of objects, essentially surjective on morphisms and fully faithful on 2-cells, we obtain effectiveness conditions for data of descent on objects, morphisms and 2-cells. It would have been hard to give the definition of 2-stack in these explicit terms from the beginning, as we would not have known the correct coherences to ask in the various gluing conditions. Our natural implicit definition is instead able to guide us in finding the right coherence conditions. Our definition of 2-stack and our characterization in terms of explicit gluing conditions have been developed in [2].

Finally, we will present the motivating example for our notion of 2-stack, which is the one of quotient 2-stack. In [1], we generalized principal bundles and quotient stacks to the categorical context of sites. We then aimed at a generalization of our theory one dimension higher, to the context of bisites, motivated by promising applications of principal 2-bundles to higher gauge theory. But there was no notion of higher dimensional stack suitable for the produced analogues of quotient prestacks in the two-categorical context. Our notion of 2-stack is able to fill this gap. Indeed, we have proven that, if the bisite satisfies some mild conditions, our analogues of quotient stacks one dimension higher are 2-stacks. Our theory of principal 2-bundles and quotient 2-stacks has been developed in [3].

Quotient 2-stacks could give a substantial contribution towards the development of a cohomology theory of schemes, and more in general of stacks, with coefficients in stacks of abelian 2-groups. This theory would produce new and refined 2-categorical invariants associated to schemes and algebraic stacks, that could solve numerous open problems in algebraic geometry.

References

[1] E. Caviglia, *Generalized principal bundles and quotient stacks*, Theory Appl. Categ. 39 (2023).

[2] E. Caviglia, 2-stack over bisites, preprint arXiv:2403.08030, 2024.

[3] E. Caviglia, Principal 2-bundles and quotient 2-stacks, preprint arXiv:2403.09379, 2024.

[4] R. Street, Characterization of bicategories of stacks, Category theory. Applications to algebra, logic and topology, Proc. int. Conf., Gummersbach 1981, Lect. Notes Math. 962, 282-291(1982).

Crossed modules of algebras over an operad and an application to rational homotopy theory

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Abstract.

Crossed modules have been studied in various contexts for a long time in algebraic topology, beginning with the work of Whitehead in [1] to understand pointed homotopy 2 types. Loday in [2] shows that crossed modules of groups can be understood in many ways : as groups internal to the category of small categories, as simplicial groups with Moore complex of length 1, as a $1-$cat-groups, or as a categories internal to the category of groups. These last two descriptions allow him to generalize crossed modules of groups to higher versions, namely n-cat-groups. Morally such an object is an n-fold category internal to groups, and higher versions of crossed module, say n-crossed modules can be inductively defined as a crossed module internal to the category of $(n-1)$-crossed modules. The other idea of Loday is that one can associate to an $n-$crossed module $\mathcal{L} = \mathbb{H} \overset{d}{\to} \mathbb{G}$ a space $B\mathcal{L}$ such that the canonical sequence $B\mathbb{H} \to B\mathbb{G} \to B\mathcal{L}$ is a homotopy fiber sequence. Using these ideas, he was able to prove that n-crossed modules are models for pointed homotopy $(n+1)$-types. Later on, several authors studied crossed modules in other algebraic contexts. For example Ellis in [3] studies n-crossed cubes of associative/commutative/Lie/etc.. algebras and proves that such objects can be equivalently defined as n-fold categories internal to algebras of the associated type. In the 00's, Janelidze in [4] gives a general framework in which crossed modules can be defined : semi-abelian categories, and proves that the category of crossed modules internal to a semi-abelian category $\mathcal{C}$ is equivalent to the category $Cat(\mathcal{C})$ of internal categories to $\mathcal{C}$. One can play the same game and prove that n-fold crossed modules internal to $\mathcal{C}$ are equivalent to n-fold categories internal to $\mathcal{C}$. Many algebraic categories are known to be semi-abelian : the categories of groups, non-unital rings, associative or commutative algebras, Lie algebras, etc ... And it seems to be folklore that the category of algebras over a symmetric algebraic reduced operad is semi-abelian.

However the definitions of Janelidze and Ellis are not so well adapted to the case of algebras over an operad. First of all, their definitions are a bit involved, Indeed they both required lots of axioms. For example, a crossed module of associative algebras is the data of a morphism of algebras $d : A \to B$, an internal action of B on A such that d is equivariant and satisfies a "Peiffer" condition. When one wants to go to higher crossed modules, this becomes awful. Second, the homotopical properties, and especially the link between (higher) crossed modules of algebras and the homotopy theory of algebras over an operad is not clear at all.

In a current work, Leray-Rivière-Wagemann (LRW) give yet another definition of crossed modules of algebras over an operad $\mathcal{P}$, namely it is the data of a $\mathcal{P}$-algebra structure on a chain complex $...0 \to A \overset{d}{\to} B$ concentrated in degrees 0 and 1. This has two main advantages over the previous

definitions. First it is a very concise and clear definition. Second it is clearly linked to the homotopy theory of $\mathcal{P}$-algebras, that is differential graded $\mathcal{P}-$algebras.

In an upcoming series of two papers, we generalize the definition of LRW to get a very concise definition of crossed n-cube of algebras over an operad, namely it is the data of a $\mathcal{P}-$algebra structure on an $n-$fold chain complex $A_{\bullet,...,\bullet}$ concentrated in degrees $(\epsilon_1, ..., \epsilon_n), \epsilon_i \in \{0,1\}$. We prove that such a structure descends to $\mathcal{P}-$algebra structures on each $A_{\epsilon_1,...,\epsilon_n}$, and that we get a crossed n-cube of algebras in the sense of Ellis. We also prove that our category of crossed n-cubes of $\mathcal{P}-$algebras is equivalent to the category of $n-$fold categories internal to the category of $\mathcal{P}-$algebras, so our definition is equivalent to the ones of Ellis and Janelidze. The main advantage of our "global" definition as opposed to the previous "locals" ones lies in the existence of the monoidal functor $Tot^{\oplus} : Ch(...Ch(\mathcal{A})) \to Ch(\mathcal{A})$ as soon as $\mathcal{A}$ is a monoidal abelian category, for example $\mathcal{A} = Vect_{\mathbb{Q}}$. In particular it induces a functor which sends a crossed $n-$cube of $\mathcal{P}$-algebras (in our sense) to a differential graded $\mathcal{P}$-algebra concentrated in degrees $0, ..., n+1$, so here the link with the homotopy theory of $\mathcal{P}$-algebras is almost for free.

The second paper of this series is devoted to an application to rational homotopy theory and especially to a conjecture of Félix and Tanré in [?]. In this paper they construct a crossed module of groups $C(\mathfrak{g})$ associated to a complete differential graded Lie algebra $\mathfrak{g}$ concentrated in degrees 0 and 1. They prove that the classifying space $BC(\mathfrak{g})$ of this crossed module is isomorphic to the geometric realization $\langle\mathfrak{g}\rangle$ of $\mathfrak{g}$ in the sense of [6] and conjecture that given a cdgl concentrated in degrees $0, ..., n$, one can associated to it a n-cat-group and that its classifying space is isomorphic to the geometric realization of this Lie algebra. We studied the case $n = 2$ and prove that for a Lie algebra in square $\mathfrak{g}$, one can associate $Tot(\mathfrak{g})$ a complete dgLie algebra of height 3 and its geometric realization $\langle Tot(\mathfrak{g})\rangle$. Or one can associate a crossed square of groups and consider its classifying space. We prove that thoses two constructions are weakly homotopy equivalent.

References

[1] A. Whitehead, *combinatorial homotopy. II*, Appl. Categ. Structures 25 (2024), no. 2, 1159–2076.

[2] J-L. Loday, *Spaces with finitely many non-trivial homotopy groups*, Journal of Pure and Applied Algebra, vol. 24, 1982, 179-202, https://doi.org/10.1016/0022-4049(82)90014-7.

[3] G-J. Ellis, *Higher dimensional crossed modules of algebras*, Journal of Pure and Applied Algebra, vol. 52, 1988, 277-282, https://doi.org/10.1016/0022-4049(88)90095-3

[4] G. Janelidze, *Internal Crossed Modules*, Georgian Mathematical Journal, vol. 10, no. 1, 2003, pp. 99-114. https://doi.org/10.1515/GMJ.2003.99

[5] Y. Félix, D. Tanré, *Realization of Lie algebras and classifying spaces of crossed modules*, Algebr. Geom. Topol. 24 (2024) 141-157, https://doi.org/10.2140/agt.2024.24.141

[6] U. Buijs, Y. Félix, A. Murillo, D. Tanré *Lie Models in Topology*, Springer Nature Switzerland AG 2020, https://doi.org/10.1007/978-3-030-54430-0

Configuration spaces of points and degenerate higher categories

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Abstract.

Degenerate higher categories are those in which a certain number of the lowest dimensions are trivial, and these structures provide a good test case for theories of n-category. Via a dimension shift, a k-degenerate $(n+k)$-category can be regarded as an n-category with extra structure, allowing us to study higher-dimensional structures via lower-dimensional ones that might be better understood.

Thus far the theory of these structures is not very well-developed, but another issue is the paucity of examples. In this talk we will present a series of examples derived from the fundamental n-groupoids of **2**, the indiscrete space with two points; here $n \leq \infty$. This may seem simple-minded (among other reasons because the space is contractible) but it provides a starting point for several interesting insights into the issues of degeneracy and commutativity.

One of the main ideas of the topic is that if the lowest k dimensions of a higher category are trivial we can disregard them, and consider the k-cells to be the 0-cells of a new, lower-dimensional structure. The k types of composition they had as k-cells become k monoidal structures on the new lower-dimensional structure, which results in the following slogan:

A k-degenerate $(n+k)$-category "is" a k-tuply monoidal n-category.

Another key idea is that the k monoidal structures can be seen to result in certain types of commutativity, via generalised Eckmann–Hilton arguments. The structures that are expected to arise are organised into the "periodic table of n-categories" conjectured in [1]. For example, a 2-degenerate 3-category "is" a 2-tuply monoidal category, which is in turn seen, via a weak Eckmann–Hilton argument, to be a braided monoidal category.

The general theory of these structures is not very rigorously understood, even in terms of the basic definitions, let alone how we perform the dimension shift and convert the compositions into monoidal structures, how the generalised Eckmann–Hilton arguments work, what structures arise from those arguments, and what "is" really means.

In this talk we work with Trimble's definition of higher category, in which composition is parametrised by operad actions. However, we will crucially use the little intervals operad $\mathcal{C}_1$ rather than the universal operad acting on path spaces as originally specified by Trimble. The definition comes with an

immediate notion of fundamental n-groupoid; here n can also be ∞, by means of the work of [2]. Our first suite of examples then is the fundamental n-groupoid of $\mathbf{2}$, for $n \leq \infty$. The k-cells are essentially continuous maps $I^k \to \mathbf{2}$, and as $\mathbf{2}$ is the subobject classifier of Top we can regard these as subspaces of k-cubes. (Here I denotes the closed interval $[0,1]$.) Composition proceeds by "stacking" the cubes in any of the k possible directions, and reparametrising via the little intervals operad.

We can then restrict our attention to the k-degenerate version, where for all $j \leq k$ the only j-cell is the empty subspace of I^j. This produces a (relatively) concrete example of k-degenerate $(n+k)$-categories, and we show how to regard this as a k-tuply monoidal n-category.

Our more specific example of interest comes from restricting our attention further, to study configurations of points in n-space. We construct an n-degenerate $(n+1)$-category derived from the fundamental ∞-groupoid of $\mathbf{2}$ as follows:

- For $j < n$ there is just one j-cell, the empty subspace of I^j.
- The n-cells are the *finite* subspaces of I^n, where all lower-dimensional boundary cells are the empty subspace; thus these amount to configurations of points in the interior of I^n.
- The $(n+1)$-cells are braids, realised as subspaces of I^{n+1}.

The properties of the little intervals operad $\mathcal{C}_1$ enable us to prove that these cells are closed under composition parametrised by $\mathcal{C}_1$. This gives us, for all $n \geq 1$, an n-fold monoidal category of configurations of points in I^n and braids between them.

This will enable us, in a sequel, to study n-degenerate $(n+1)$-categories in generality, and, following our work in [3, 4], exhibit a biequivalence between a suitable 2-category of n-degenerate $(n+1)$-categories and the 2-category of symmetric monoidal categories.

References

[1] John Baez and James Dolan. Higher-dimensional algebra and topological quantum field theory. *Jour. Math. Phys.*, 36:6073–6105, 1995.

[2] Eugenia Cheng and Tom Leinster. Weak infinity-categories via terminal coalgebras. *Theory and Applications of Categories*, 34:1073–1133, 2019.

[3] Eugenia Cheng and Alexander S. Corner. Weak vertical composition To appear in *Theory and Applications of Categories*, 2022

[4] Eugenia Cheng and Alexander S. Corner. Weak vertical composition II: totalities To appear in *Theory and Applications of Categories*, 2023

The smothering model structure on **Cat**

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Abstract.

The notion of smothering functor introduced by Riehl and Verity ([1]) defines a class of functors which extends the usual notion of equivalence of categories by relaxing the faithfulness condition. These functors appear naturally as comparison functors between homotopy categories, for instance the comparison functor relating the homotopy category of a pullback of quasicategories to the pullback of the homotopy categories of these quasicategories.

In this work, we consider a slight strengthening of the original definition of smothering functors, which we refer to as stably smothering, that actually form the trivial fibrations of a right Bousfield localization of the natural model structure on **Cat**. Precisely, the generating cofibration are given by the inclusion i_k (for $k \geq 0$) of two opposite face of the k-cube in the k-cube as well as i_c. Those are pictured for $0 \leq k \leq 2$ on the left below. The generating trivial cofibration is j, as on the right below

$$\begin{array}{c} \emptyset \\ \downarrow{\scriptstyle i_0} \\ \bullet \end{array} \qquad \begin{array}{c} \bullet \quad \bullet \\ \downarrow{\scriptstyle i_1} \\ \bullet \to \bullet \end{array} \qquad \begin{array}{c} \bullet \to \bullet \\ \bullet \to \bullet \\ \downarrow{\scriptstyle i_2} \\ \begin{array}{ccc} \bullet & \to & \bullet \\ \downarrow & & \downarrow \\ \bullet & \to & \bullet \end{array} \end{array} \qquad \begin{array}{c} \bullet \to \bullet \\ \downarrow{\scriptstyle i_c} \\ \bullet \simeq \bullet \end{array} \qquad\qquad \begin{array}{c} \bullet \\ \downarrow{\scriptstyle j} \\ \bullet \simeq \bullet \end{array}$$

The weak equivalences for this model structure are the weakly stably smothering functors. They can be defined either as the functors which are essentially surjective on objects and lift against all the generating cofibration, except possibly i_0, or equivalently as the functors having the 2-categorical right lifting property against the generating cofibration. Unlike weakly smothering functors in the sense of [1], they enjoy the 2-ouf-of-3 property, which is crucial to prove the following:

Theorem 1. *There is a cofibrantly generated model structure on* **Cat** *with weak equivalences the weakly stably smothering functors.*

In parallel, we introduce an equivalence relation, *indiscernibility*, on parallel isomorphisms in a given category C. The starting point is an attempt to quotient out all parallel isomorphisms as to discard the structure provided by a given isomorphism $\alpha : x \simeq y$ between two objects, and only keep the mere existence of such an isomorphism between x and y. It turns out that such an identification is not necessarily compatible with the categorical structure of C. To make this statement precise, we consider the simplicial object $\mathbf{core}(C_\bullet^\to)$ in **Gpd** whose groupoid of n-simplices is that of paths of arrows in C of length n and isomorphisms between two such paths. Given a relation R on parallel isomorphism, there is a quotient $|\mathbf{core}(C_\bullet^\to)|_R$ whose groupoid of n-simplices has arrows the equivalence class of

isomorphisms between paths of length n in C (with respect to the pointwise equivalence relation on isomorphisms between two given paths deduced from R). While this simplicial object need not satisfy the Segal condition for an arbitrary relation, there exists an equivalence relation $\mathcal{R}$ on parallel isomorphisms which can be characterized as the maximal relation R such as the simplicial object $|\mathbf{core}(C_\bullet^{\rightarrow})|_R$ yields a category object in $\mathbf{Gpd}$. One can then observe that weakly smothering functors preserve and reflect indiscernibility and admit the following characterization:

Proposition 2. *A functor $F : C \to D$ is weakly smothering if and only if it preserves indiscernibility and the induced transformation $|\mathbf{core}(C_\bullet^{\rightarrow})|_{\mathcal{R}_C} \to |\mathbf{core}(D_\bullet^{\rightarrow})|_{\mathcal{R}_D}$ is a pointwise equivalence of groupoids.*

The indiscernibility relation $\mathcal{R}$ can be extended to a relation $\mathcal{R}'$ between parallel arrows which is a congruence with respect to composition of arrows, and the canonical quotient map $C \to \Pi(C)$ is a smothering functor, where $\Pi(C)$ is the category with the same object as C and with morphisms the indiscernibility classes of arrows modulo $\mathcal{R}'$. The weakly smothering functors are equivalently those inducing an equivalence of categories $\Pi(C) \to \Pi(D)$.

The following example, adapted from Proposition 3.3.14 of [2], is archetypical:

Example 3.
Given a pullback squares of quasicategories as on the right with p an isofibration, the canonical functor

$$Ho(A \times_B E) \to Ho(A) \times_{Ho(B)} Ho(E)$$

is a weakly stably smothering functor.

$$\begin{tikzcd}
A \times_B E \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

This can in fact be account for the following important result, which states that, in the adjunction between the homotopy category functor $\mathbf{Ho}$ and the nerve functor $\mathbf{N}$, the left and right adjoints are swapped when restricting to the ∞-coreflection of $\mathcal{C}at$ given by the smothering model structure:

Theorem 4. *There is a diagram of ∞-adjunctions*

$$\begin{tikzcd}
\mathcal{C}at_{\boldsymbol{smt}} \arrow[rr, bend left, "\mathbf{N}\circ\iota_s"] \arrow[rr, phantom, "\perp"] \arrow[dr, bend left, "\iota_s"] \arrow[dr, phantom, "\dashv"] & & \mathcal{Q}\mathcal{C}at \arrow[ll, bend left, "\tau_s\circ\mathbf{Ho}"] \arrow[dl, bend left, "\mathbf{Ho}"] \arrow[dl, phantom, "\dashv"] \\
& \mathcal{C}at \arrow[ul, bend left, "\tau_s"] \arrow[ur, bend left, "\mathbf{N}"] &
\end{tikzcd}$$

where $\mathcal{C}at_{\boldsymbol{smt}}$, $\mathcal{C}at$ and $\mathcal{Q}\mathcal{C}at$ are the $(\infty,1)$-categories presented by the smothering model structure, the natural model structure on $\mathbf{Cat}$ and the Joyal model structure on simplicial sets respectively.

References

[1] E. Riehl and D. Verity,*Elements of ∞-Category Theory*, Cambridge University Press, vol. 194, 2022.

[2] E. Riehl and D. Verity, *The 2-category theory of quasi-categories*, Advances in Mathematics 25 (2024), vol. 280, 549–642.

Differential bundles in Goodwillie calculus

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Abstract.

In recent work with Kristine Bauer and Matthew Burke [1] we developed the theory of *tangent ∞-categories*, the ∞-categorical version of Rosický's [9] and Cockett and Cruttwell's [3] theory of tangent categories. This theory allows us to make precise the analogy between Goodwillie's calculus of functors in homotopy theory [6] and the ordinary differential calculus of smooth manifolds, by constructing a tangent ∞-category whose objects are ∞-categories and whose morphisms are functors. The tangent bundle on an ∞-category $\mathcal{C}$ is that constructed by Lurie [7]: the ∞-category $T\mathcal{C}$ of *parameterized spectra* (in the sense of stable homotopy theory) over objects of $\mathcal{C}$.

Construction of the Goodwillie tangent structure opens the door for extending other ideas from tangent categories, and hence from smooth manifolds, to the functor calculus setting. In this talk, I will describe joint work with Kaya Arro in which we establish the analogues of smooth vector bundles. Cockett and Cruttwell [5] developed a notion of *differential bundle* in an arbitrary tangent category, and MacAdam showed in [8] that in the category of smooth manifolds that notion recovers precisely the smooth vector bundles (of locally constant rank). Our main result identifies a differential bundle in the Goodwillie tangent structure with a collection of stable ∞-categories and exact functors between them, parameterized by a base ∞-category.

Our work comprises three parts. First we extend Cockett and Cruttwell's definition of differential bundle to the setting of tangent ∞-categories. Then we show how any differential bundle can be recovered, up to equivalence, from its projection map and zero section by appropriate pullbacks. (This characterization is inspired by MacAdam's work but appears to be new, even for ordinary tangent categories.) Finally, we apply that construction to the Goodwillie tangent structure and establish our classification of differential bundles (and linear maps between them) in that setting.

Given the prominent role that vector bundles play in the theory of manifolds, we expect differential bundles to be central to the tangent category perspective on Goodwillie calculus. For example, we hope our framework will allow for a concrete definition of the *cotangent* bundle on an ∞-category. Concepts such as connections, torsion and curvature [4] and Lie algebroids [2] have also been defined in an arbitrary tangent category, and we expect these notions could now be identified in the world of ∞-categories too.

Full Schedule
16:00 - Tuesday

References

[1] K. Bauer, M. Burke, and M. Ching, *Tangent ∞-categories and Goodwillie calculus*, (2023), arxiv:2101.07819v4.

[2] M. Burke and B. MacAdam, *Involution algebroids: a generalisation of Lie algebroids for tangent categories*, 2019, arxiv:1904.06594.

[3] J. R. B. Cockett and G. S. H. Cruttwell, *Differential structure, tangent structure, and SDG*, Appl. Categ. Structures **22** (2014), no. 2, 331–417.

[4] J. R. B. Cockett and G. S. H. Cruttwell, *Connections in tangent categories*, Theory Appl. Categ. **32** (2017), Paper No. 26, 835–888.

[5] J. R. B. Cockett and G. S. H. Cruttwell, *Differential bundles and fibrations for tangent categories*, Cah. Topol. Géom. Différ. Catég. **59** (2018), no. 1, 10–92.

[6] T. G. Goodwillie, *Calculus. III. Taylor series*, Geom. Topol. **7** (2003), 645–711 (electronic).

[7] J. Lurie, *Higher Algebra*, (2017), available at `www.math.ias.edu/~lurie/`.

[8] B. MacAdam, *Vector bundles and differential bundles in the category of smooth manifolds*, Appl. Categ. Structures **29** (2021), no. 2, 285–310.

[9] J. Rosický, *Abstract tangent functors*, Diagrammes **12** (1984), JR1–JR11.

Fibered elementary quotient completion

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Abstract.

Completions of categories by quotients have been deeply studied in category theory. The main construction is the free exact completion of a (weakly) left exact category provided in [1, 2]. Later in [3], Maietti and Rosolini introduced the elementary quotient completion in order to give an abstract description of the *quotient model* in [4]. The main novelty is that the authors relativize the notions of equivalence relation and quotient for Lawvere's elementary doctrines, which are suitable functors of the form $\mathsf{P} : \mathbb{C}^{\mathrm{op}} \to \mathsf{Pos}$, from a category $\mathbb{C}$ with strict finite products to the category Pos of posets and order preserving functions. As shown in [3], the elementary quotient completion generalizes the exact completion of (weakly) left exact categories.

The present work originates from the following observation: the exact completion of a left exact category $\mathbb{C}$ not only adds stable quotients of equivalence relations; it also provides *fibered* quotients with respect to the codomain fibration, in the sense that there exists a fibered adjunction $Q \dashv \mathrm{Eq}$ as in the following diagram

$$\begin{tikzcd}
\mathrm{EqRel}(\mathbb{C}^{\to}_{ex/lex}) \arrow[rr, bend left, "Q"] \arrow[rr, phantom, "\perp"] \arrow[dr, "\mathrm{r}"'] & & \mathbb{C}^{\to}_{ex/lex} \arrow[ll, "\mathrm{Eq}"] \arrow[dl, "\mathrm{cod}"] \\
& \mathbb{C}_{ex/lex} &
\end{tikzcd}$$

where the left hand fibration is that of equivalence relations on $\mathbb{C}^{\to}_{ex/lex}$, i.e. congruences $r_1, r_2 : z \to x$ in $\mathbb{C}^{\to}_{ex/lex}$ and $\mathrm{r}(r_1, r_2) := \mathrm{cod}(x)$, and Eq is the functor sending an object $x \in \mathbb{C}^{\to}_{ex/lex}$ to the pair (id_x, id_x). Pointwisely, the above diagram states that the slices of $\mathbb{C}_{ex/lex}$ have quotients of equivalence relations, which is a consequence of the fact the slices of $\mathbb{C}_{ex/lex}$ are also exact.

In this talk, we aim to generalize the above situation and to freely add fibered quotients with respect to an elementary doctrine $\mathsf{P} : \mathbb{C}^{\mathrm{op}} \to \mathsf{Pos}$ for a fibered category $p : \mathbb{E} \to \mathbb{C}$ with the same base category. To this extent, we assume that p is a comprehension category as in [5] in order to take into account P-equivalence relations on objects of $\mathbb{E}$ given by the subfibration on equivalence relations $\mathrm{r} : \mathrm{EqRel}_{\mathbb{E}}(\mathsf{P}) \to \mathbb{C}$ of the fibration of relations obtained through the following change of base situation

$$\begin{tikzcd}
& \mathrm{Rel}_{\mathbb{E}} \arrow[r] \arrow[d] \arrow[dl, dashed, "\mathrm{r}"'] \arrow[dr, phantom, "\lrcorner"] & \int \mathsf{P} \arrow[d, "\pi(\mathsf{P})"] \\
\mathbb{C} & \mathbb{E} \arrow[l, "p"'] \arrow[r, "A \to \Gamma.A.A"'] & \mathbb{C}
\end{tikzcd}$$

where $\Gamma.A$ is the common notation for the domain of the comprehension of an element $A \in \mathbb{E}$ over Γ, and where $\pi(\mathsf{P})$ is the Grothendieck construction applied to P.

Starting from a suitable pair of fibrations $(\pi(\mathsf{P}), p)$ where P is an elementary doctrine and p is a comprehension category, we provide a pair $(\pi(\overline{\mathsf{P}}), \overline{p})$ where $\overline{\mathsf{P}} : \overline{\mathbb{C}}^{\mathrm{op}} \to \mathsf{Pos}$ is the elementary quotient completion of P and $\overline{p}$ is a comprehension category with equality and fibered quotients as defined in [5], i.e. there exists a fibered adjunction as in the following diagram

$$\begin{tikzcd}
\mathrm{EqRel}_{\overline{\mathbb{E}}}(\overline{\mathsf{P}}) \arrow[dr, "\mathrm{r}"'] \arrow[rr, bend left, "Q"] \arrow[rr, phantom, "\perp"] & & \overline{\mathbb{E}} \arrow[ll, "\mathrm{Eq}"] \arrow[dl, "\overline{p}"] \\
& \overline{\mathbb{C}} &
\end{tikzcd}$$

$\overline{p}$ is obtained considering suitable *pseudo descent objects*, in the sense of [6], and the correspondence $(\pi(\mathsf{P}), p) \mapsto (\pi(\overline{\mathsf{P}}), \overline{p})$ has a suitable universal property in 2-categorical terms.

As applications of this work, we provide an abstract method for models of *families of sets* for the extensional level of the *Minimalist foundation* [7], as done in [8] for the *Predicative effective topos* introduced in [9].

References

[1] A. Carboni and R. Celia Magno, *The free exact category on a left exact one*, Journal of the Australian Mathematical Society (1982), 33(3), pp.295-301.

[2] A. Carboni and E.M. Vitale, *Regular and exact completions.* J. Pure Appl. Algebra 125.1-3 (1998).

[3] M.E. Maietti and G. Rosolini, *Quotient completion for the foundation of constructive mathematics*, Logica Universalis (2013).

[4] M.E. Maietti, *A minimalist two-level foundation for constructive mathemathics*, Ann. Pure Appl. Logic 160.3 (2009), pp. 319–354.

[5] B. Jacobs, *Categorical logic and type theory*, Studies in Logic and the Foundations of Mathematics, vol. 141,North-Holland Publishing Co., Amsterdam, (1998).

[6] C. Hermida *Descent on 2-Fibrations and Strongly 2-Regular 2-Categories*, Appl. Categ. Structures 12 (2004), no. 5-6.

[7] M.E. Maietti and G. Sambin, *Toward a minimalistic foundation for constructive mathematics*, From sets and types to topology and analysis (2005).

[8] C.J. Cioffo, S. Maschio and M.E. Maietti, *Fibred sets within a predicative variant of Hyland's effective topos*, In preparation.

[9] S. Maschio and M.E. Maietti, *A predicative variant of Hyland's effective topos*, The Journal of Symbolic Logic 86.2 (2021), 433-447.

The algebraic weak factorisation system of twisted coreflections and delta lenses

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Abstract.

The notion of an algebraic weak factorisation system (AWFS), introduced by Grandis and Tholen [5], generalises that of an orthogonal factorisation system (OFS). In the standard definition of an AWFS on a category $\mathcal{C}$, the left and right classes of morphisms are determined by the categories of L-coalgebras and R-algebras for a suitable comonad-monad pair (L, R) on the arrow category $\mathcal{C}^{\mathbf{2}}$. An OFS may be understood as an AWFS in which the comonad and monad are idempotent. Recently, Bourke [2] demonstrated that an AWFS on $\mathcal{C}$ can be defined entirely in terms of a pair of double categories $\mathbb{L}$ and $\mathbb{R}$ over the double category of commutative squares $\mathbb{S}\mathrm{q}(\mathcal{C})$ equipped with a *lifting operation* that satisfies two axioms, yielding a characterisation much closer in spirit to that of an OFS.

A leading example of an algebraic weak factorisation system is the AWFS on $\mathcal{C}\mathrm{at}$ whose L-coalgebras are the split coreflections (functors equipped with a right-adjoint-left-inverse) and whose R-algebras are the split opfibrations. Delta lenses are functors equipped with a functorial choice of lifts, directly generalising the notion of split opfibration [6], and are the focus of ongoing research in applied category theory. Motivated by this close relationship with split opfibrations, it is natural to ask: is there an AWFS on $\mathcal{C}\mathrm{at}$ whose R-algebras are delta lenses?

In this talk, I will introduce the notion of *twisted coreflection* as a split coreflection with a certain property, and construct an algebraic weak factorisation system on $\mathcal{C}\mathrm{at}$ whose L-coalgebras are the twisted coreflections and whose R-algebras are the delta lenses.

I will present two separate descriptions of this AWFS, highlighting the connections between them. In the first approach [3], I will define explicitly the comonad L and monad R arising from a functorial factorisation on $\mathcal{C}\mathrm{at}$, in the sense of Grandis and Tholen [5]. In the second approach [4], I will construct double categories $\mathbb{T}\mathrm{wCoref}$ and $\mathbb{L}\mathrm{ens}$ of twisted coreflections and delta lenses, respectively, and define suitable lifts of twisted coreflections against delta lenses, in the sense of Bourke [2]. Both methods make important use the universal properties of bijective-on-objects functors, discrete categories, and the comprehensive factorisation system on $\mathcal{C}\mathrm{at}$. Moreover, I will show that this AWFS is cofibrantly generated by a small double category, in the sense of Bourke and Garner [1].

In addition to providing a new framework for understanding delta lenses, one of the principal benefits of this work is illustrating a seemingly rare example of a cofibrantly generated AWFS in which the entire left class, not just the generators, may be fully understood. In particular, I will show that every twisted coreflection arises as a pushout of an initial functor from a discrete category along a bijective-on-objects functor. This yields a simple way of constructing examples of twisted coreflections from indexed collections of categories with a chosen initial object. I will also give an explicit construction of the cofree twisted coreflection on a functor *and* on a split coreflection.

References

[1] J. Bourke & R. Garner, *Algebraic weak factorisation systems I: Accessible AWFS*, Journal of Pure and Applied Algebra 220 (2016).

[2] J. Bourke, *An orthogonal approach to algebraic weak factorisation systems*, Journal of Pure and Applied Algebra 227 (2023).

[3] B. Clarke, *The algebraic weak factorisation system for delta lenses*, Electronic Proceedings in Theoretical Computer Science 397 (2023).

[4] B. Clarke, *Lifting twisted coreflections against delta lenses*, preprint arXiv:2401.17250, 2024.

[5] M. Grandis & W. Tholen, *Natural weak factorization systems*, Archivum Mathematicum 42 (2006).

[6] M. Johnson & R. Rosebrugh, *Delta lenses and opfibrations*, Electronic Communications of the EASST 57 (2013).

Drazin Inverses in Categories

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Abstract.

A Drazin inverse [1] is a fundamental algebraic gadget which has been extensively deployed in semigroup theory and ring theory. While they can also be defined for endomorphisms of any category, Drazin inverses (it seems) have never been extensively developed from a categorical perspective [4]. The purpose of this talk is to introduce Drazin inverses and to present some of their basic results. A Drazin category is a category in which every endomorphism has a Drazin inverse: examples include the category of matrices over a field, the category of finite length modules over any ring, and any finite set enriched category. We shall discuss Drazin inverses in mere categories, in additive categories, and in dagger categories. We shall explain how Drazin inverses relate to Leinster's notion of eventual image duality [2]. Finally we shall introduce a new notion of Drazin inverses for pairs of opposing maps, and as an application of this kind of Drazin inverse we provide, for dagger categories, a novel characterization of the Moore-Penrose inverse [3] in terms of the Drazin inverse of the opposing pair of a map and its adjoint.

References

[1] M. P. Drazin. Pseudo-inverses in associative rings and semigroups. *The American mathematical monthly*, 65(7):506?514, 1958.

[2] T. Leinster. The eventual image. *arXiv preprint arXiv:2210.00302*, 2022.

[3] J.R.B. Cockett and J-S. P. Lemay. Moore-Penrose Dagger Categories. In Proceedings of the Twentieth International Conference on *Quantum Physics and Logic*, Paris, France, 17-21st July 2023, volume 384 of *Electronic Proceedings in Theoretical Computer Science*, pages 171-186. Open Publishing Association, 2023.

[4] J.R.B. Cockett, J-S. P. Lemay, and P. Varshinee Srinivasn. Drazin Inverses in Categories. *arXiv preprint arXiv:2402.18226*, 2024

A higher-dimensional Eckmann–Hilton argument

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Abstract. The Eckmann–Hilton argument plays a subtle and crucial role behind the scenes in higher-dimensional category theory. It can show us where hidden commutativities arise from cells with identity boundaries being able to commute past each other, given enough dimensions. The basic case tells us that doubly-degenerate 2-categories give commutative monoids; more precisely, a doubly-degenerate 2-category "is" a set with two multiplications on it satisfying interchange, and the Eckmann–Hilton argument tells us that those multiplications must be the same and commutative [3].

Given more dimensions, more nuance is possible. It is considered well-known that doubly-degenerate weak 3-categories "are" categories with two weak monoidal structures satisfying weak interchange, and that a weak Eckmann–Hilton argument shows that this amounts to a braided monoidal category [4]. However, the classic proofs of this do not explicitly provide the generalisation of the Eckmann–Hilton argument [5].

At the next dimension the situation seems even more folkloric. It seems widely accepted that a category with three monoidal structures on it (satisfying appropriate interchange) "is" a symmetric monoidal category, and that this is the largest number of monoidal structures that will fit: adding further monoidal categories just gives us symmetric monoidal categories again. Often no indication of how this is achieved is given, only a reference made to Baez and Dolan's Stabilisation Hypothesis [2] or an appeal to the work of Joyal and Street [5].

All of this is in some sense well known, but we do not that think this has been precisely written down. Aguiar and Mahajan [1] give a comprehensive account of the related notion of n-monoidal category, but this still does not quite fit the particular nuance that we are interested in: that multiple monoidal structures, inherited from higher compositions and so interacting appropriately under interchange, will 'stabilise' to what is effectively a single symmetric monoidal structure.

In another sense, the work of Joyal and Street [5] claims that identifying those braidings which are symmetries provides an equivalence of 2-categories between symmetric monoidal categories and categories with a symmetric multiplication, but the details to establish this equivalence are not described explicitly.

Often an appeal is made to the Eckmann–Hilton argument in order to fill in missing details, but often nothing resembling such an argument is provided in full. In this talk we will give the 3-fold generalisation of the weak Eckmann–Hilton argument in 4 dimensions, that is, for 3-degenerate 4-categories. This will enable us to not only give a precise sense in which 3-degenerate 4-categories give rise to symmetric monoidal categories, but also to fill in the details of [5] to provide:

1. a generalisation to n-degenerate $(n+1)$-categories,
2. an algebraically nice 2-category of these, and
3. a proof that this 2-category is biequivalent to the 2-category of symmetric monoidal categories.

This enables us to finally remove the quotation marks around "is" in the statement

An n-degenerate $(n+1)$-category "is" a symmetric monoidal category

to give a satisfying, fully algebraic proof of one part of the stabilisation hypothesis.

References

[1] Marcelo Aguiar and Swapneel Mahajan. Monoidal functors, species and hopf algebras. *CRM Monograph Series*, 2010.

[2] John C Baez and James Dolan. Higher-dimensional algebra and topological quantum field theory. *Journal of mathematical physics*, 36(11):6073–6105, 1995.

[3] Beno Eckmann and Peter J Hilton. Group-like structures in general categories I multiplications and comultiplications. *Mathematische Annalen*, 145(3):227–255, 1962.

[4] Robert Gordon, Anthony John Power, and Ross Street. *Coherence for tricategories*, volume 558. American Mathematical Soc., 1995.

[5] André Joyal and Ross Street. Braided tensor categories. *Advances in Mathematics*, 102(1):20–78, 1993.

Connections in algebraic geometry via tangent categories

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Abstract.
In this talk, we'll use the abstract formalism of tangent categories [1, 5] to compare and contrast different notions of connections in algebraic geometry. In particular, we'll show (i) how the definition of a "connection on a differential bundle in a tangent category" [2], when applied to the tangent category of affine schemes [3], exactly corresponds to the definition of a connection on a module [4, pg. 756], and (ii) how the definition of a "connection on a submersion in a tangent category", when applied to the tangent category of affine schemes, generalizes connections on a module, and seems to be a new concept in algebraic geometry.

This talk is based on joint work with JS Lemay and Eli Vandenburg (for i) and Marcello Lanfranchi (for ii).

References

[1] J. R. B. Cockett and G. S. H. Cruttwell, *Differential structure, tangent structure, and SDG*, Appl. Categ. Structures **22** (2014), no. 2, 331–417.

[2] J. R. B. Cockett and G. S. H. Cruttwell, *Connections in tangent categories*, Theory and Applications of Categories, **32** (2023), no. 26, 835–888.

[3] G. S. H. Cruttwell and JS Lemay, *Differential Bundles in Commutative Algebra and Algebraic Geometry*, Theory and Applications of Categories, **39** (2023), no. 36, 1077–1120.

[4] S. Lang, *Algebra*, Graduate Texts in Mathematics, 1993.

[5] J. Rosický, *Abstract tangent functors*, Diagrammes **12** (1984), JR1–JR11.

Non-additive derived functors: a chain complex approach

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Abstract.

Let $F\colon \mathcal{C} \to \mathcal{E}$ be a functor from a category $\mathcal{C}$ to a (Borceux–Bourn [1]) homological or (Janelidze–Márki–Tholen [6]) semi-abelian category $\mathcal{E}$. We investigate conditions for the homology $\mathrm{H}_n(X, F)$ of an object X in $\mathcal{C}$ with coefficients in the functor F defined via projective resolutions in $\mathcal{C}$ to be independent of the chosen resolution. Then the left derived functors of F may be constructed as in the classical abelian case.

Our strategy is to extend the concept of *chain homotopy* to a non-additive setting via the technique of *imaginary morphisms*. More precisely, we use the *approximate subtractions* of Bourn–Janelidze [2], originally considered in the context of subtractive categories [7, 8]. This works as soon as $\mathcal{C}$ is a pointed regular category with finite coproducts and enough projectives which are *closed under protosplit subobjects*, a new condition we introduce in [3], and which comes for free in the abelian setting. We further assume that the functor F satisfies certain exactness conditions: we may ask it to be protoadditive [4, 5] and preserve binary coproducts and proper morphisms, for instance—conditions which amount to F being additive when $\mathcal{C}$ and $\mathcal{E}$ are abelian categories.

In this setting we work out a basic theory of derived functors, compare it with the simplicial approach, and give some examples.

The main reference of this talk is [3].

References

[1] F. Borceux and D. Bourn, *Mal'cev, protomodular, homological and semi-abelian categories*, Math. Appl., vol. 566, Kluwer Acad. Publ., 2004.

[2] D. Bourn and Z. Janelidze, *Subtractive categories and extended subtractions*, Appl. Categ. Structures 17 (2009), 317–343.

[3] M. Culot, F. Renault and T. Van der Linden, *Non-additive derived functors*, in preparation, 2024.

[4] T. Everaert and M. Gran, *Homology of n-fold groupoids*, Theory Appl. Categ. 23 (2010), no. 2, 22–41

[5] T. Everaert and M. Gran, *Protoadditive functors, derived torsion theories and homology*, J. Pure Appl. Algebra 219 (2015), no. 8, 3629–3676.

[6] G. Janelidze, L. Márki, and W. Tholen, *Semi-abelian categories*, J. Pure Appl. Algebra 168 (2002), no. 2–3, 367–386.

[7] Z. Janelidze, *Subtractive categories*, Appl. Categ. Structures 13 (2005), 343–350.

[8] A. Ursini, *On subtractive varieties I*, Algebra Universalis 31 (1994), 204–222.

Two developments of "Separability of The Second Kind"

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Abstract. The categorical notion of a separable functor was first given by Năstăsescu, Van den Bergh, and Van Oystaeyen in [12]. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is said to be separable if the natural transformation on hom-sets induced by F can be split by a natural transformation P. This definition is constructed so that separable morphisms of rings correspond to the restriction of scalars being separable in the sense of [12]. Therefore, the study of separable functors is closely tied to that of adjoint pairs. In [13], Rafael gave conditions in terms of the unit and counit of an adjunction (F, G) for the functors F or G to be separable. This was further generalized to the notion of separability of the second kind, by Caenepeel and Militaru [5] : Suppose that we have functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $I : \mathcal{C} \longrightarrow \mathcal{X}$. Then, the functor F is said to be I-separable if the natural transformation on hom-sets induced by F is split up to the natural transformation induced by I.

This talk will consist of two parts, both revolving around the notion of separability of the second kind. The first part is published work with Abhishek Banerjee [2], in which we bring together separability of the second kind and another notion of separable functors that has appeared in the literature : Heavily separable functors due to Ardizzoni and Menini [1] i.e., separable functors F such that the splitting natural transformation P is compatible with compositions in $\mathcal{D}$ in a certain manner. We combine these ideas to consider functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ which are **heavily I-separable**, where I is a functor $I : \mathcal{C} \longrightarrow \mathcal{X}$.

We proceed to give a Rafael-type Theorem appropriate for this "amalgamated" version of separability. This characterizes heavy separability of the second kind for functors admitting a (left or right) adjoint. We then present applications of these results in three different contexts. The first application is in the context of ringoids, which horizontally categorify rings. These results generalize Ardizzoni and Menini's results on the ordinary heavy separability of functors associated with ring extensions.

The second application is to monads and comonads and the associated Eilenberg-Moore adjunctions. By fixing a category $\mathcal{C}$ and a monad $\mathbf{T}$ on $\mathcal{C}$, one can look at the family of $\mathbf{T}$-adjunctions, i.e., adjoint pairs $(F : \mathcal{C} \longrightarrow \mathcal{D}, G : \mathcal{D} \longrightarrow \mathcal{C})$ whose associated monad is $\mathbf{T}$. If $I : \mathcal{C} \longrightarrow \mathcal{X}$ is any functor and (F, G), (F', G') are $\mathbf{T}$-adjunctions, we show [2, § 4.] that the left adjoint F is heavily I-separable if and only if so is F'. This means that for a given monad $\mathbf{T}$, we can ask if the family of $\mathbf{T}$-adjunctions as a whole, is heavily I-separable. A dual result holds for comonads. These results are motivated by the work of Mesablishvili [10] with I-separability and families of adjunctions associated to a given monad or comonad. Now suppose that (L, R) is an adjunction such that the left adjoint L can be equipped with the structure of a comonad $\mathbf{L}$ on $\mathcal{C}$. It is known ([3, § 2.6]) that the right adjoint R

can be equipped with the structure of a monad $\mathbf{R}$. For any adjoint pair $(I \dashv J)$ of endofunctors on $\mathcal{C}$ satisfying a commutativity condition [2, Theorem 4.3.], we see that the free $\mathbf{L}$-coalgebra functor $F^{\mathbf{L}}$ taking objects of $\mathcal{C}$ to free $\mathbf{L}$-coalgebras is heavily I-separable if and only if the functor $F_{\mathbf{R}}$ taking objects of $\mathcal{C}$ to free $\mathbf{R}$-algebras is heavily J-separable. We then combine this with the results of Ardizzoni and Menini [1] to give two applications.

If time permits, we also see a third application of the Rafael-type Theorem in the context of entwined modules. This is one of the original contexts studied by Caenepeel and Militaru while introducing separability of the second kind in [5].

In the second part of the talk, we enrich the notion of separability of the second kind over a symmetric monoidal closed category $\mathcal{V}$. In particular, using Lawvere's remarkable idea of viewing metric spaces as enriched categories, we see that separability of the second kind yields a very simple geometric condtion when $\mathcal{V} = (([0,\infty], \geq), +, 0)$. We end with two theorems : First, we see an enriched version of the Rafael-type Theorem appropriate to this context. Second, we see that this notion of enriched separability is invariant under change of base.

References

[1] A. Ardizzoni, C. Menini, *Heavily separable functors*, J. Algebra 543 (2020), 170–197.

[2] A. Banerjee, S. Das, *Heavily separable functors of the second kind and applications*, J. Algebra 643 (2024), 49–75.

[3] G. Böhm, T.Brzeziński, R. Wisbauer, *Monads and comonads on module categories*, J. Algebra 322 (2009), 1719–1747.

[4] T. Brzeziński, *Frobenius properties and Maschke-type theorems for entwined modules*, Proc. Amer. Math. Soc. 128 (2000), 2261–2270.

[5] S. Caenepeel, G. Militaru, *Maschke functors, semisimple functors and separable functors of the second kind: applications*, J. Pure Appl. Algebra 178 (2003), 131–157.

[6] S. Eilenberg, J. C. Moore, *Adjoint functors and triples*, Illinois J. Math. 9 (1965), 381–398.

[7] S. Estrada, S. Virili, *Cartesian modules over representations of small categories*, Adv. Math. 310 (2017), 557–609.

[8] G. M. Kelly, *Basic concepts of enriched category theory*, London Mathematical Society Lecture Note Series, vol. 64, Cambridge University Press, 1982.

[9] G. M. Kelly, R. Street, *Review of the elements of 2-categories*, Lecture Notes in Math. vol. 420, Springer, 1974.

[10] B. Mesablishvili, *Monads of effective descent type and comonadicity*, Theory Appl. Categ. 16 (2006), 1–45.

[11] B. Mitchell, *Rings with several objects*, Advances in Math. 8 (1972), 1–161.

[12] C. Năstăsescu, M. Van den Bergh, F. Van Oystaeyen, *Separable functors applied to graded rings*, J. Algebra 123 (1989), 397–413.

[13] M. D. Rafael, *Separable functors revisited*, Comm. Algebra 18 (1990), 1445–1459.

First-Order Bicategories: a new categorical perspective on first-order logic

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Abstract.

Background Cartesian bicategories (of relations) were introduced in [2] by Carboni and Walters as a categorical algebra of relations. These are poset-enriched categories such that every object X is a special Frobenius bimonoid, comonoids are left adjoint to the monoids and every arrow is a lax comonoid homomorphism.

The archetypical example is the category $\mathbf{Rel}^{\circ}$, whose objects are sets, arrows are relations, composition is the relational product $R \,\mathring{;}\, S = \{(x,y) \mid \exists z.(x,z) \in R \wedge (z,y) \in S\}$, identities are $id^{\circ}_X = \{(x,y) \mid x = y\}$ and the order on the arrows is given by inclusion.

A category $\mathbf{C}$ is a cocartesian bicategory if $\mathbf{C}^{\text{co}}$, i.e. $\mathbf{C}$ with the order reversed, is a cartesian bicategory. An example is $\mathbf{Rel}^{\bullet}$, the category of sets and relations, whose composition is the relative sum $R \,\bullet\!\!;\, S = \{(x,y) \mid \forall z.(x,z) \in R \vee (z,y) \in S\}$, identities are $id^{\bullet}_X = \{(x,y) \mid x \neq y\}$ and the order on the arrows is inclusion.

The axioms of cartesian bicategories are known to be complete for regular logic [6], that is the $\exists\wedge$-fragment of first-order logic (FOL). Dually, cocartesian bicategories are complete for coregular logic, i.e. the $\forall\vee$-fragment of FOL.

In order to account for full first-order (classical) logic, we investigate the interaction of cartesian and cocartesian bicategories as linear bicategories.

The latter were introduced in [3] by Cockett, Koslowski and Seely as a horizontal categorification of linearly distributive categories and roughly consists of two bicategories sharing the same objects and morphisms but having two diffrent compositions, one linearly distributing over the other.

Rel is an example and, indeed, the following holds for every Q, R, S of the appropriate type:

$$Q \fatsemi (R \bullet S) \subseteq (Q \fatsemi R) \bullet S$$

Main contributions We introduce the novel notion of *first-order bicategories* as linear bicategories where the two bicategory structures are cartesian and cocartesian, moreover:

1. there are *linear adjunctions* [3] between (co)monoids of the two structures;

2. and these are required to satisfy some *linear Frobenius* conditions.

In the spirit of functorial semantics, we take the free first-order bicategory $\mathbf{FOB}_{\mathbb{T}}$ generated by a theory $\mathbb{T}$ and observe that models of $\mathbb{T}$ in a first-order bicategory $\mathbf{C}$ are morphisms $\mathcal{M}\colon \mathbf{FOB}_{\mathbb{T}} \to \mathbf{C}$. Taking $\mathbf{C} = \mathbf{Rel}$, these are models in the sense of FOL.

By adapting Henkin's proof of Gödel's completeness theorem, we prove that the laws of first-order bicategories provide a complete axiomatisation for first-order logic.

Then, we proceed in showing a correpondence between first-order bicategories and boolean hyperdoctrines, reusing from [1] an adjunction between cartesian bicategories and *elementary and existential doctrines* [5], which are a generalisation of hyperdoctrines, corresponding to regular logic.

Our result reveals an adjunction between the category of first-order bicategories and the category of boolean hyperdoctrines.

Leveraging another result from [1], we demonstrate that the adjunction becomes an equivalence when restricted to well-behaved hyperdoctrines (i.e. those whose equality is extentional and satisfying the rule of unique choice [5]). Finally, combining this finding with a result in [5], we illustrate that *functionally complete* [2] first-order bicategories are equivalent to boolean categories [4].

References

[1] Filippo Bonchi, Alessio Santamaria, Jens Seeber, and Paweł Sobociński. On doctrines and cartesian bicategories. In *9th Conference on Algebra and Coalgebra in Computer Science (CALCO 2021)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2021.

[2] A. Carboni and R. F. C. Walters. Cartesian bicategories I. *Journal of Pure and Applied Algebra*, 49(1):11–32, November 1987.

[3] J. Robin B. Cockett, Jürgen Koslowski, and Robert AG Seely. Introduction to linear bicategories. *Mathematical Structures in Computer Science*, 10(2):165–203, 2000.

[4] P.T. Johnstone. *Sketches of an elephant: a topos theory compendium*, volume 2 of *Studies in Logic and the foundations of mathematics*. Oxford Univ. Press, 2002.

[5] M.E. Maietti, F. Pasquali, and G. Rosolini. Triposes, exact completions, and hilbert's ε-operator. *Tbilisi Mathematica journal*, 10(3):141–166, 2017.

[6] Jens Seeber. *Logical completeness for string diagrams*. PhD thesis, 2020.

Axioms for the category of finite-dimensional Hilbert spaces and linear contractions

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Abstract.
The category **Hilb** of Hilbert spaces and bounded linear maps and the category **Con** of Hilbert spaces and linear contractions were both recently characterised in terms of simple category-theoretic structures and properties [2, 3]. For example, the structure of a *dagger*—an involutive identity-on-objects contravariant endofunctor—encodes adjoints of linear maps. Remarkably, none of the axioms refer to analytic notions such as norms, continuity or real numbers.

Counterintuitively, characterising categories with only *finite-dimensional* Hilbert spaces is more challenging than those with *all* Hilbert spaces. The problem is that directed colimits are the natural categorical way to encode analytic completeness of the scalar field, but the existence of too many of these colimits also implies the existence of objects corresponding to infinite-dimensional spaces. In fact, to prove that the scalars are the real or complex numbers without such an infinite-dimensional object, appeal to Solèr's theorem [4] is no longer possible, so an entirely new approach is necessary.

This talk will introduce a characterisation, stated below, of the category **FCon** of *finite-dimensional* Hilbert spaces and linear contractions. It will focus on a new approach to proving that the scalars are the real or complex numbers that does not rely on the existence of infinite-dimensional objects. In this proof, the supremum of a bounded increasing sequence of positive scalars is explicitly constructed using the colimit of a directed diagram associated to the sequence. The talk will also briefly touch on the new notions of *bounded sequential diagram* and *dagger finiteness*, which are needed to address finite dimensionality. It is based on recent joint work with Chris Heunen [1].

Theorem. *A locally small dagger rig category* $(\mathbf{D}, \otimes, I, \oplus, O)$ *is equivalent to* **FCon** *if and only if*

(1) the object O is terminal (and thus a zero object), the canonical projections

$$p_1 = \big(X \oplus Y \xrightarrow{1 \oplus 0} X \oplus O \cong X\big) \qquad \textit{and} \qquad p_2 = \big(X \oplus Y \xrightarrow{0 \oplus 1} O \oplus Y \cong Y\big)$$

are jointly monic, and there is a morphism $d\colon I \to I \oplus I$ such that $p_1 d \neq 0 \neq p_2 d$;

(2) the object I is dagger simple and a monoidal separator;

(3) every parallel pair has a dagger equaliser and every dagger monomorphism is a kernel;

(4) for all epimorphisms $x\colon A \to X$ and $y\colon A \to Y$, we have $x^\dagger x = y^\dagger y$ if and only if there is an isomorphism $f\colon X \to Y$ such that $y = fx$;

(5) every bounded sequential diagram has a colimit; and,

(6) every object is dagger finite.

References

[1] M. Di Meglio and C. Heunen. "Dagger categories and the complex numbers: Axioms for the category of finite-dimensional Hilbert spaces and linear contractions". 2024. arXiv preprint: 2211.02688.

[2] C. Heunen and A. Kornell. "Axioms for the category of Hilbert spaces". In: *Proceedings of the National Academy of Sciences* 119.9 (2022), e2117024119. DOI: 10.1073/pnas.2117024119.

[3] C. Heunen, A. Kornell, and N. van der Schaaf. "Axioms for the category of Hilbert spaces and linear contractions". In: *Bulletin of the London Mathematical Society* (2024). DOI: 10.1112/blms.13010.

[4] M. P. Solèr. "Characterization of Hilbert Spaces by Orthomodular Spaces". In: *Communications in Algebra* 1 (1995), pp. 219–243. DOI: 10.1080/00927879508825218.

Pushforward monads

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Abstract.

If $\mathbf{T}$ is a monad on $\mathcal{C}$ and $G\colon \mathcal{C} \to \mathcal{D}$ is right adjoint to F, then $G\mathbf{T}F$ is a monad on $\mathcal{D}$, which we denote $G_\#\mathbf{T}$. Even when G is not a right adjoint, we can define $G_\#\mathbf{T}$ subject to the existence of a right Kan extension. This is the pushforward of $\mathbf{T}$ along G. This construction was first considered by Street in [2], where it takes place in a general (strict) 2-category, but has received very little attention since.

In this talk, I will review its definition, introduce its functoriality properties with respect to G and $\mathbf{T}$, and state the universal property satisfied by $G_\#\mathbf{T}$. Pushforwards turn out to be intimately related to codensity monads: pushing the identity monad forward along G gives the codensity monad of G. Moreover, any pushforward monad is a codensity monad, whereby $G_\#\mathbf{T}$ is the codensity monad of $GU^{\mathbf{T}}$, with $U^{\mathbf{T}}$ being the forgetful functor from the category of $\mathbf{T}$-algebras.

I will then present examples of the pushforward of three families of monads on the category of finite sets along the inclusion $\mathbf{FinSet} \hookrightarrow \mathbf{Set}$. Each of these turn out to be related to the well-known codensity monad of this inclusion, which was shown to be the ultrafilter monad by Kennison and Gildenhuys [1]. Lastly, I will identify the category of algebras of the codensity monad of $\mathbf{Field} \hookrightarrow \mathbf{Ring}$ as the free product completion of $\mathbf{Field}$, denoted $\mathbf{Prod}(\mathbf{Field})$. Pushing this monad forward along the forgetful functor $\mathbf{Ring} \to \mathbf{Set}$ gives the codensity monad of $\mathbf{Field} \to \mathbf{Set}$. Its category of algebras is still $\mathbf{Prod}(\mathbf{Field})$, giving the remarkable fact that $\mathbf{Prod}(\mathbf{Field})$ is monadic over $\mathbf{Set}$, even though $\mathbf{Field}$ is famously not.

References

[1] J. F. Kennison, and D. Gildenhuys, *Equational completion, model induced triples and pro-objects*, Journal of Pure and Applied Algebra 1.4 (1971).

[2] R. Street, *The formal theory of monads*, Journal of Pure and Applied Algebra 2.2 (1972).

The homotopy theory of Eilenberg-Zilber opetopic sets

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Abstract.

We present the construction of the category of Eilenberg-Zilber presheaves (EZ-presheaves for short), which is a coherent reflective subcategory of the category of presheaves on a Reedy category satisfying certain natural axioms (we call such Reedy category a tidy Reedy category).

This construction is motivated by the fact that the category of EZ-presheaves allows the use of techniques familiar from the theory of simplicial sets, such as induction over skeleta. In general, such techniques are not directly applicable to all presheaves. In fact, in the case of the category Δ, the category of EZ-presheaves is equal to the category of all presheaves, the fact which is not true for many other Reedy categories of interest. Consequently, some key properties of that construction are described.

Next, we apply our construction to the category $pOpe_\iota$ of positive opetopes with face maps and ι-contractions, introduced and studied by Zawadowski ([2]). As a preparation, we show that $pOpe_\iota$ is a tidy Reedy category. A minor modification of Olschok's theorem ([1]) allows us to endow this category $\widehat{pOpe_\iota}_{EZ}$ of opetopic EZ-presheaves with a model structure in a Cisinski style, which we call the opetopic $(\infty, 0)$-structure.

Finally, we construct two adjunctions between the categories $\widehat{pOpe_\iota}_{EZ}$ and $sSet$ and show that they are in fact Quillen equivalences (when the category of simplicial sets is considered with the Kan-Quillen model structure), by studying properties of certain fundamental opetopic sets, called opetopic associahedra (which are the images of representable simplicial sets in both of the adjunctions mentioned above).

References

[1] M. Olschok, *Left determined model structures for locally presentable categories*, Appl. Categ. Structures 19 (2011)

[2] M. Zawadowski, *Positive Opetopes with Contractions form a Test Category*, ArXiv:1712.06033 [math.CT], (2017).

Iterating semidirect products in semi-abelian categories

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Abstract.

In any semi-abelian category $\mathcal{C}$, the notions of semidirect product, internal action and split epimorphisms are equivalent [1]. In particular, this means that every split short exact sequence is, up to isomorphism, of the form

$$A \xrightarrow{j_A} A \rtimes B \underset{s_B}{\overset{p_B}{\rightleftarrows}} B.$$

Furthermore, the notion of action of B on A is internal, in the sense that it can be defined as a morphism $B\flat A \to A$ such that certain diagrams commute in $\mathcal{C}$. In the categories of groups and Lie algebras, a notion of n-semidirect products has been introduced by Carrasco and Cegarra [2, 3], which allows to construct certain iterated semi-direct products using a system of functions $A_k \times A_j \to A_i$ for $i \leq j < k$ satisfying certain identities.

In this talk we will explain how these n-semidirect products can be characterized by diagrams linking various exact sequences. We will also show how they can be constructed as colimits in $\mathcal{C}$, and how their structure can be made internal, i.e described by morphisms and commutative diagrams in $\mathcal{C}$, and how that description can be simplified when $\mathcal{C}$ is algebraically coherent [4].

We will also show how iterated semidirect products can be related to iterated internal actions, and use this relation to explain when the iteration is associative.

References

[1] Borceux, F.; Janelidze, G. ; Kelly, G. Internal object actions. *Commentationes Mathematicae Universitatis Carolinae*, 46 (2005), no. 2, 235–255

[2] Carrasco, P. ; Cegarra, A. Group-theoretic algebraic models for homotopy types. *Journal Of Pure And Applied Algebra*. 75 (1991), no. 3 , 195–235

[3] Carrasco, P. ; Cegarra, A. A Dold-Kan theorem for simplicial Lie algebras. *Theory And Applications Of Categories*, 32 (2017), no. 34, 1165–1212

[4] Cigoli, A.; Gray, J. ; Van der Linden, T. Algebraically coherent categories. *Theory And Applications Of Categories*, 30 (2015), no. 54, 1864-1905

Galois theory and homology in quasi-abelian functor categories

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Abstract.

In this presentation based on [2], I will consider the category $\mathscr{A}^{\mathbb{T}}$ of functors from a finite category $\mathbb{T}$ to a quasi-abelian category $\mathscr{A}$, and show that, for any replete full subcategory $\mathbb{S}$ of $\mathbb{T}$, the full subcategory $\mathscr{F}$ of $\mathscr{A}^{\mathbb{T}}$ with objects the functors $F : \mathbb{T} \to \mathscr{A}$ with $F(T) = 0$ for all $T \notin \mathbb{S}$ is a reflective and, moreover, torsion-free subcateogry of $\mathscr{A}^{\mathbb{T}}$. This implies that the corresponding Galois structure is admissible, and I will characterize the (higher) central extensions in $\mathscr{A}^{\mathbb{T}}$ with respect to $\mathscr{F}$ and the classes of regular epimorphisms in $\mathscr{A}^{\mathbb{T}}$ and $\mathscr{F}$, respectively. More precisely, for a regular epimorphism α in $\mathscr{A}^{\mathbb{T}}$, the following conditions are equivalent:

1. α is a central extension.
2. The kernel $\mathrm{Ker}(\alpha)$ of α lies in $\mathscr{F}$.
3. The T-component α_T is an isomorphism for all $T \notin \mathbb{S}$.

Furthermore, I will give generalized Hopf formulae for homology.

Instances of the pair $(\mathscr{A}^{\mathbb{T}}, \mathscr{F})$ are given by $(\mathrm{Arr}(\mathscr{A}), \mathscr{A})$, $(\mathrm{Arr}^2(\mathscr{A}), 2\text{-}\mathrm{Arr}(\mathscr{A}))$ and, more generally, $(\mathrm{Arr}^n(\mathscr{A}), n\text{-}\mathrm{Arr}(\mathscr{A}))$ for every $n \geq 1$, where $n\text{-}\mathrm{Arr}(\mathscr{A})$ denotes the category with objects the chain complexes in $\mathscr{A}$ of length n. Since $\mathscr{A}$ is assumed to be quasi-abelian, $\mathrm{Arr}^n(\mathscr{A})$ is equivalent to the category $\mathrm{Grpd}^n(\mathscr{A})$ of internal n-fold groupoids in $\mathscr{A}$ and $n\text{-}\mathrm{Arr}(\mathscr{A})$ is equivalent to the category $n\text{-}\mathrm{Grpd}(\mathscr{A})$ of internal n-groupoids in $\mathscr{A}$.

Let me shortly recall the notions of (higher) central extensions and generalized Hopf formulae for homology.

Categorical Galois theory

Categorical Galois theory, see e.g. [3], generalizes both classical Galois theory and the theory of central extensions of groups. A *Galois structure* Γ consists of an adjunction

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift left, "\mathsf{F}"] \arrow[r, phantom, "\perp"] & \mathscr{F} \arrow[l, shift left, "\mathsf{U}"] \end{tikzcd}$$

with unit η, and classes $\mathscr{E}$ and $\mathscr{Z}$ of morphisms in $\mathscr{C}$ and $\mathscr{F}$, respectively, that satisfy certain conditions. For any object B in $\mathscr{C}$, this induces an adjunction

$$\begin{tikzcd} \mathscr{E}(B) \arrow[r, shift left, "\mathsf{F}^B"] \arrow[r, phantom, "\perp"] & \mathscr{Z}(\mathsf{F}(B)), \arrow[l, shift left, "\mathsf{U}^B"] \end{tikzcd}$$

where $\mathscr{E}(B)$, also denoted by $\mathrm{Ext}(B)$, is the full subcategory of the slice category $\mathscr{C} \downarrow B$ with objects the morphisms in $\mathscr{E}$ with codomain B. These are called the *extensions* of B. Categorical Galois theory is concerned with the study of the full subcategory $\mathrm{CExt}(B)$ of $\mathrm{Ext}(B)$ with objects the central extensions of B. This notion is defined in two steps:

- An extension $f : A \to B$ is called *trivial* if it lies in the essential image of U^B.
- It is called *central* if it is 'locally' trivial, i.e., there exists a *monadic extension* $p : E \to B$ such that the pullback $p^*(f)$ of f along p is a trivial extension.

If the Galois structure Γ is *admissible*, the fundamental theorem of categorical Galois theory asserts that, for any monadic extension $p : E \to B$, there is a characterization of the extensions of B, whose pullback along p is a trivial extension, in terms of internal actions of the *Galois pregroupoid* $\mathrm{Gal}(E, p)$.

The central extensions with respect to the Galois structure Γ_{Ab} given by the adjunction

$$\begin{tikzcd} \mathrm{Grp} \arrow[r, shift left, "\mathrm{Ab}"] \arrow[r, phantom, "\perp"] & \mathrm{Ab}, \arrow[l, shift left, "\mathrm{I}"] \end{tikzcd}$$

where Ab and I are the abelianization and inclusion functors, respectively, and $\mathscr{E}$ and $\mathscr{Z}$ are the classes of surjective group homomorphisms in Grp and Ab, respectively, recover exactly the classical central extension of groups.

<u>Generalized Hopf formulae for homology</u>

In certain cases, see e.g. [1], the full subcategory $\mathrm{CExt}(\mathscr{C})$ of central extensions in $\mathscr{C}$ of $\mathrm{Ext}(\mathscr{C})$ induces itself a Galois structure Γ_1 with adjunction

$$\begin{tikzcd} \mathrm{Ext}(\mathscr{C}) \arrow[r, shift left, "\mathsf{F}_1"] \arrow[r, phantom, "\perp"] & \mathrm{CExt}(\mathscr{C}) \arrow[l, shift left, "\mathsf{U}_1"] \end{tikzcd}$$

with unit η^1, and with $\mathscr{E}^1$ and $\mathscr{Z}^1$ the classes of *double extensions* defined relatively to $\mathscr{E}$ and $\mathscr{Z}$, respectively. It turns out that the functor $[-]^1 : \mathrm{Ext}(\mathscr{C}) \to \mathrm{Ext}(\mathscr{C})$ given on objects by $[f] := \mathrm{Ker}(\eta^1_f)$, factors through $\mathscr{C}$, i.e., there exists a functor $[-]_1 : \mathrm{Ext}(\mathscr{C}) \to \mathscr{C}$ such that $[-]^1 = \iota^1 \circ [-]_1$, where ι^1 maps an object B to the extension $B \to 0$.

If $p : P \to B$ is a $\mathscr{E}$*-projective presentation* of B, the *second Hopf formula for homology* of B with respect to $\mathscr{F}$ is defined as

$$H_2(B, \mathscr{F}) := \tfrac{[P] \cap \mathrm{Ker}(p)}{[p]_1},$$

where $[P] := \mathrm{Ker}(\eta_P)$. More generally, the $(n+1)$-st Hopf formula for homology $H_{n+1}(B, \mathscr{F})$ is defined using the notions of n-fold central extensions and n-fold $\mathscr{E}$-projective presentations.

The generalized Hopf formulae with respect to the Galois structure Γ_{Ab} recover exactly the integral homology groups.

References

[1] T. Everaert, M. Gran and T. Van der Linden, *Higher Hopf formulae for Homology via Galois Theory*, Adv. in Mathematics 217 (2008), 2231–2267.

[2] N. Egner, *Galois theory and homology in quasi-abelian functor categories*, preprint arXiv:2403.12750, 2024.

[3] G. Janelidze, *Categorical Galois theory: revision and some recent developments*, Galois connections and applications, Math. Appl., vol. 565, Kluwer Acad. Publ., 2004, 139–171.

Gray multicategories and left and right Gray skew-multicategories

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Abstract.
Let $\mathcal{O}_n$ be the category $(n\text{-}\mathrm{Cat})\text{-}\mathrm{Cat}$ of categories enriched in n-categories, or $\mathrm{Cat}(n\text{-}\mathrm{Cat})$, the category of categories internal in n-categories, with $n \geq 1$. We introduce *Gray multicategory* and *left and right Gray skew-multicategory* of $\mathcal{O}_n$. They all differ by *type* $(*, \bullet)$ which depends on the nature $*$ of functors and $\bullet$ of transformations used in their construction. We show (in law dimensions) that left Gray skew-multicategories of certain $\bullet_1$-type are left closed and left representable, whereas right Gray skew-multicategories of other $\bullet_2$-type are right closed and right representable. For Gray multicategories we show that: 1) those of strict $*$-type are representable, and 2) that those of certain $\bullet_1$-type are left closed, and those of other $\bullet_2$-type are right closed, both for any $*$-type, and that they are related by duality. Using the results of Hermida on the equivalence of representable multicategories and monoidal categories, and of Bourke-Lack that left representable skew-multicategories yield skew-monoidal categories, we obtain non-Cartesian monoidal and skew-monoidal (closed) categories of internal and enriched categories depending on the type. Our construction generalizes results of Gray and Böhm of closedness and Gray-monoidality for the categories of 2-categories and double categories (in the sense of dimension and type, respectively), and recent results of Bourke-Lobbia for Gray-categories (in the sense of type). The approach we use relies on the tools developed in [4] and [5].

This is a work in progress, the results are expected not to depend on dimension n.

References

[1] G. Böhm, *The Gray Monoidal Product of Double Categories*, Appl. Categ. Structures 28 (2020), 477–515.

[2] J. Bourke, S. Lack, *Skew monoidal categories and skew multicategories*, J. Algebra 56 (2018), 237–266.

[3] J. Bourke, G. Lobbia: *A skew approach to enrichment for Gray-categories*, Adv. Math. 434 (2023).

[4] B. Femić: *Bifunctor Theorem and strictification tensor product for double categories with lax double functors*, Theory Appl. Categ. 39 (2023), no. 29, 824–873.

[5] B. Femić, *Enrichment and internalization in tricategories, the case of tensor categories and alternative notion to intercategories*, Filomat 38, (2004), no. 8, 2601–2660.

[6] J. W. Gray, *Formal category theory: adjointness for 2-categories*, Lecture Notes in Mathematics, vol. 391, Springer-Verlag, Berlin-New York (1974) 1, 19, 27.

[7] C. Hermida, *Representable multicategories*, Adv. Math. 151 (2000), no. 2, 164–225.

Universal central extensions of braided crossed modules of groups

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Abstract.
The concept of central extension of groups is highly relevant in mathematics, for instance, in the interpretation of the third cohomology, and it plays a fundamental role in several areas of physics as well in the quantization of symmetries.

Crossed modules of groups are algebraic objects equivalent to strict 2-groups, or equivalently categorical groups. Since crossed modules of groups are a generalization of groups, it is natural to search extensions of classical results in the theory of groups in the category of crossed modules of groups, both examples of semi-abelian categories [3].

Joyal and Street defined in [4] the concept of braiding for monoidal categories as a natural isomorphism $\tau_{X,Y}\colon X \otimes Y \to Y \otimes X$, generalizing the idea of the usual tensor product of vector spaces. The notion of braiding for categorical groups provides an equivalent category to the category of braided crossed modules of groups (see [2, 4]).

In this talk, we will devise a braided version of the results given by Norrie in [5] for braided crossed modules of groups in the framework of a semi-abelian category (see[1]); more precisely, we will study universal central extensions in the category of braided crossed modules.

For that purpose, we will construct the universal central extension of a braided crossed module in the category of braided crossed modules. Then, we will also give a canonical braiding on the universal central extension of a crossed module with a given braiding in the category of crossed modules. To finish, we will show the relationship between the two constructions.

References

[1] J. M. Casas, T. Van der Linden, Universal central extensions in semi-abelian categories, Appl. Categ. Struct. 22 (1) (2014) 253–268.

[2] D. Conduché, Modules croisés généralisés de longueur 2, J. Pure Appl. Algebra 34 (2-3) (1984) 155–178.

[3] G. Janelidze, L. Márki, W. Tholen, Semi-abelian categories, J. Pure Appl. Algebra 168 (2-3) (2002) 367–386.

[4] A. Joyal, R. Street, Braided tensor categories, Adv. Math. 102 (1) (1993) 20–78.

[5] K. J. Norrie, Crossed modules and analogues of group theorems, PhD thesis, King's College, University of London, 1987.

An Algebraic Combinatorial Approach to the Abstract Syntax of Opetopic Structures

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Abstract.
The starting point of the talk will be the identification of structure common to tree-like combinatorial objects, exemplifying the situation with abstract syntax trees (as used in formal languages) and with opetopes (as used in higher-dimensional algebra). The emerging mathematical structure will be then formalised in a categorical setting, unifying the algebraic aspects of the theory of abstract syntax [2, 3] and the theory of opetopes [6]. This realization allows one to transport viewpoints between the two mathematical theories and I will explore it here in the direction of higher-dimensional algebra, giving an algebraic combinatorial framework for a generalisation of the slice construction [1] for generating opetopes. The technical work will involve setting up a microcosm principle for near-semirings [5] and exploiting it in the bicategory of generalised species of structures [4], the cartesian closed structure of which plays a fundamental role.

References

[1] J. Baez and J. Dolan. Higher-Dimensional Algebra III. n-Categories and the Algebra of Opetopes. Advances in Mathematics 135(2):145–206, 1998.

[2] M. Fiore, G. Plotkin and D. Turi. Abstract syntax and variable binding. In Proceedings of the 14th Annual IEEE Symposium on Logic in Computer Science (LICS'99), pages 193–202. IEEE, Computer Society Press, 1999.

[3] M. Fiore. Second-order and dependently-sorted abstract syntax. In Proceedings of the 23rd Annual IEEE Symposium on Logic in Computer Science (LICS'08), pages 57–68. IEEE, Computer Society Press, 2008.

[4] M. Fiore, N. Gambino, M. Hyland, and G. Winskel. The cartesian closed bicategory of generalised species of structures. J. London Math. Soc., 77:203–220, 2008.

[5] M. Fiore and P. Saville. List objects with algebraic structure. In Proceedings of the 2nd International Conference on Formal Structures for Computation and Deduction (FSCD 2017), No. 16, pages 1–18, 2017.

[6] S. Szawiel and M. Zawadowski. The web monoid and opetopic sets. Journal of Pure and Applied Algebra, 217:1105–1140, 2013.

Monoidal Meta-Theorem

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Abstract.

Certain families of theories of multi-sorted universal algebra can be modeled in monoidal, symmetric monoidal, and cartesian monoidal categories, respectively. For each of these families of theories, we produce a sound deduction system $\vdash$. We show that these deduction systems are complete with respect to the cartesian monoidal category of sets. This yields a meta-theorem:

Let C be a (cartesian/symmetric) monoidal category and let $E \cup \{\phi\}$ be a (cartesian/symmetric) monoidal theory of universal algebra. Then

$$E \models \phi \textit{ in } \mathbf{Set} \textit{ implies that } E \models \phi \textit{ in } C.$$

The Monoidal Meta-Theorem makes a modest connection between the algebraic structures in **Set** to monoidally enriched algebraic structures. As a corollary, we attain that the Eckmann-Hilton argument generalizes to the setting of symmetric monoidal categories.

The meta-theorem for cartesian monoidal categories is essentially known and proven in [1].

References

[1] P. T. Johnstone, *Sketches of an Elephant: A Topos Theory Compendium: Volume 2*, Oxford Logic Guides, Clarendon Press, 2002.

Conditional flatness and fiberwise localizations in semi-abelian categories

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Abstract. In [1] we extended the group-theoretic notion of *conditional flatness* for a localization functor to any pointed category, and we investigated it in the context of homological and semi-abelian categories. This context includes several examples of importance in algebra, such as groups, loops, Lie algebras, crossed modules, C*-algebras, etc. In the presence of *functorial fiberwise localization*, many results analogous to those obtained in the category of groups [2] hold in any semi-abelian category, and we provide some existence theorems for certain localization functors in some specific categories, such as groups, compact groups and cocommutative Hopf algebras. Among reflective subcategories, the so-called *Birkhoff subcategories* are the ones that are also closed in the larger category under subobjects and quotients. We prove that any Birkhoff subcategory of a semi-abelian category provides a conditionally flat localization, and explain how the property of conditional flatness of a functor actually corresponds to the property of admissibility of an adjunction from the point of view of categorical Galois theory [3].

We then present a new example of Birkhoff subcategory [4]. When $\mathbb{C}$ is a regular Mal'tsev category, hence in particular if $\mathbb{C}$ is semi-abelian, the category 2-$\mathsf{Grpd}(\mathbb{C})$ of internal 2-groupoids in $\mathbb{C}$ can be shown to be a Birkhoff subcategory of the category $\mathsf{Grpd}^2(\mathbb{C})$ of double groupoids in $\mathbb{C}$, and a simple description of the reflector can be given. Under some natural conditions on a semi-abelian category $\mathbb{C}$ first considered in [5], the semi-abelian category 2-$\mathsf{Grpd}(\mathbb{C})$ turns out to be also action representable in the sense of [6].

These results have been obtained in collaboration with Jérôme Scherer [1] and Nadja Egner [4].

References

[1] M. Gran, J. Scherer, *Conditional flatness, fiberwise localizations, and admissible reflections*, J. Australian Math. Society. 116 (2024) 200-220.

[2] E.D. Farjoun, J. Scherer, *Conditionally flat functors on spaces and groups*, Collectanea Mathematica 66 (2015) 149-160.

[3] G. Janelidze, *Pure Galois Theory in Categories*, J. Algebra 132 (1990) 270-286.

[4] N. Egner and M. Gran, *Double groupoids and 2-groupoids in a regular Mal'tsev category*, preprint (in preparation).

[5] M. Gran and J. R. Gray, *Action representability of the category of internal groupoids*, Theory Appl. Categories 37 (2021) 1-13.

[6] F. Borceux, G. Janelidze, and G. M. Kelly, *Internal object actions*, Comment. Math. Univ. Carolin. 46 (2005) 235-255.

From Kripke models to neighborhood models in category theory

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Abstract.

There is a well-known one-to-one correspondence between Kripke models and augmented neighbourhood models, where an augmented model is a monotonic neighbourhood model which contains its core ($\bigcap N(w) \in N(w)$ for all w), (see [4], original proof in [1]). Axiom K is valid in Kripke models, but not in neighborhood models in general. If we consider the two categories $\mathcal{KM}$ and $\mathcal{NM}$, we can set an injective functor $f : \mathcal{KM} \to \mathcal{NM}$ that sends every Kripke model to a modally equivalent (augmented) neighbourhood model; alternatively, $\mathcal{KM}$ is isomorphic to the subcategory of augmented neighbourhood models, in which axiom K is valid. If we consider the quotient categories of bisimilar Kripke models $\mathcal{KM}\backslash bisim$ and neighbourhood models $\mathcal{NM}\backslash bisim$, we can establish an analogue functor f' between them preserving modal equivalence. Considering the injective functors $i_k : \mathcal{KM} \to \mathcal{KM}\backslash bisim$ and $i_n : \mathcal{NM} \to \mathcal{NM}\backslash bisim$, we have that $i_n \circ f = f' \circ i_k$. This can also be studied among specific kinds of models (reflexive, transitive, and so on).

Coalgebras over the category *Set* allow us to abstract transition structures like Kripke and neighbourhood frames and models, and also other structures such as labelled transition systems and deterministic automata (see [3], [5]). Thus, the aforementioned relationship among Kripke and neighborhood models can be studied in a more general perspective. Coalgebras offer an abstract framework that can be applied to generalise well-known notions from Kripke frames and models such as bisimilarity or image-finiteness for broader families of neighbourhood models and frames [2].

References

[1] B.F. Chellas, *Modal Logic - An Introduction*, Cambridge University Press, 1980.

[2] H.H. Hansen; C. Kupke; E. Pacuit, Neighbourhood structures: Bisimilarity and basic model theory. *Logical Methods in Computer Science*, 2009, vol. 5.

[3] Kupke, C., & Pattinson, Coalgebraic semantics of modal logics: An overview. *Theoretical Computer Science*, 412(38), 2011, pp. 5070-5094.

[4] E. Pacuit, *Neighborhood semantics for modal logic*, Springer, 2017.

[5] Y. Venema, Algebras and coalgebras. In *Studies in Logic and Practical Reasoning*, Vol. 3, Elsevier, 2007, pp. 331-426

Pasting diagrams beyond acyclicity

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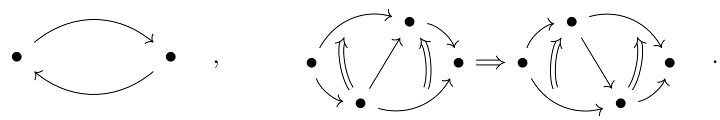
Abstract.
Most formalisms for n-categorical diagrams [1, 2, 4, 3] include some form of *acyclicity* condition on the shapes of diagrams. These conditions may be as strong as requiring the "flow" between cells across all dimensions to be acyclic, or as weak as only forbidding direct cycles when pasting along a single dimension. In either case, they forbid very simple non-pasting shapes already in dimension 1, and commonly occurring pasting shapes starting from dimension 3, such as



Moreover, acyclic shapes tend to not be stable under various useful constructions: typically, the stronger conditions are not stable under arbitrary duals, and the weaker conditions are not stable under pasting or under Gray products. A much better-behaved condition on shapes of diagrams is *regularity*: roughly, the requirement that all boundaries of all cells occurring in the diagram be, topologically, closed balls of the appropriate dimension.

The reason for the focus on acyclicity across all these sources is the insistence that the cells of the "presented ω-category" be *subsets* of cells of the diagram shape. In this talk, I will show that the problem disappears if one takes a functorial perspective, focussing on more general morphisms of diagram shapes. In particular, we can put ourselves in a convenient category $\mathbf{RDCpx}_{\downarrow}$ of *regular directed complexes*, which are poset-like structures encoding diagram shapes satisfying the regularity condition. Among the regular directed complexes, there is a class of objects, the *molecules*, which are shapes of pasting diagrams and satisfy a pasting theorem. Morphisms of $\mathbf{RDCpx}_{\downarrow}$ can be interpreted as "cellular" functors which are allowed to decrease the dimension of a cell; we let $\mathbf{RDCpx}_{=}$ be the wide subcategory whose morphisms are dimension-preserving.

The main result then states: *If P is a regular directed complex, the set $\mathit{Mol}/_P$ of isomorphism classes of objects $[f\colon U \to P]$ in the slice category $\mathbf{RDCpx}_{=}/_P$ where U is a molecule admits a natural structure of strict ω-category. Moreover, this ω-category has a minimal generating set whose cells are in bijection with the elements of P.*

If, in addition, P satisfies a technical condition called *having frame-acyclic molecules*, then $\mathit{Mol}/_P$ is a polygraph (or computad). This is a much weaker acyclicity condition which, in particular, is always satisfied in dimension ≤ 3. Since regular directed complexes are closed under all sorts of useful operations, such as gluings (pushouts of monomorphisms), all duals, Gray products, suspensions, and joins, they provide a more convenient framework for n-categorical diagrams.

The content of this talk is based on parts of the upcoming monograph [5].

References

[1] M. Johnson. The combinatorics of n-categorical pasting. *Journal of Pure and Applied Algebra*, 62(3):211–225, 1989.

[2] R. Street. Parity complexes. *Cahiers de topologie et géométrie différentielle catégoriques*, 32(4):315–343, 1991.

[3] S. Forest. Unifying notions of pasting diagrams. *Higher Structures*, 6(1):1–79, 2022.

[4] R. Steiner. The algebra of directed complexes. *Applied Categorical Structures*, 1(3):247–284, 1993.

[5] A. Hadzihasanovic. Combinatorics of higher-categorical diagrams. In preparation, 2024.

A relative comonad associated to the category of partial comodules

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Abstract. Partial comodules over a Hopf algebra can be seen as a generalization of usual comodules: the coaction should no longer be coassociative, but only *partially* coassociative. This setting is dual to the one of partial modules over a Hopf algebra, which were introduced in [2] and generalize partial actions of groups to Hopf algebras.

The category of partial comodules does not satisfy the so-called fundamental theorem of comodules: a partial comodule is not guaranteed to be equal to the sum of its finite-dimensional subcomodules. Furthermore, in general there does not even exist a coalgebra whose category of comodules is equivalent to the category of partial comodules. This is in contrast to the dual theory of partial modules; it is known that the category of partial modules over a Hopf algebra is equivalent to the category of modules over a suitably constructed algebra which moreover has the structure of a Hopf algebroid. However, partial comodules are comonadic over vector spaces, as was shown in [3].

In order to better study the structure of this comonad (e. g., could it be lifted to a Hopf monad on some monoidal category?), it is useful to look at an associated *relative comonad*. The notion of relative comonad is dual to relative monads as introduced in [1]. In our case, the comonad is relative to the inclusion functor of vector spaces into complete topological vector spaces, and it induces a comonad on this last category.

References

[1] T. Altenkirch, J. Chapman, T. Uustalu, *Monads need not be endofunctors*, Log. Methods Comput. Sci. 11 (2015), no. 1, 1–40.

[2] M. M. S. Alves, E. Batista, J. Vercruysse, *Partial representations of Hopf algebras*, J. Algebra 426 (2015), 137–187.

[3] E. Batista, W. Hautekiet, J. Vercruysse, *A comonadicity theorem for partial comodules*, Quaestiones Mathematicae, https://doi.org/10.2989/16073606.2024.2322011, to appear.

Virtual double categories as coloured box operads

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Abstract.

Virtual double categories, also considered by Leinster as fc-multicategories [1], are a 2-categorification of multicategories and compare to double categories as multicategories compare to monoidal categories [2]. In algebraic topology, multicategories are also known as coloured operads and are extensively used to encode algebraic operations, thus generalizing operads.

In this talk we will generalize operads to box operads [3], fitting the following scheme:

$$\begin{array}{ccccc} \text{operads} & \subset & \text{multicategories} = \text{coloured operads} & \supset & \text{monoidal categories} \\ \cap & & \cap & & \cap \\ \text{box operads} & \subset & \text{virtual double categories} = \text{coloured box operads} & \supset & \text{double categories} \end{array}$$

In particular, box operads correspond to virtual double categories with a single object, a single horizontal arrow and a single generating vertical arrow. We apply box operads to algebraic geometry and topology, as we now will explain.

We use a linear box operad Lax to encode lax functors over a small category taking values in linear categories. This is motivated by algebraic geometry where lax functors appear as prestacks generalizing structure sheaves and (noncommutative) deformations thereof.

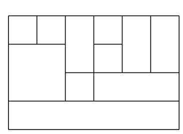
Our main results are the following:

1. The first operadic approach to an L_∞-structure on the Gerstenhaber-Schack complex of a general prestack was given in [4]. Using box operads, in [3], we give explicit formulas in terms of stackings of rectangles ("boxes").

2. Making use of a newly developed *Koszul duality for box operads* which deals with non-quadratic relations, in [5], we establish a minimal (in particular cofibrant) model Lax_∞ of Lax, shedding a new light on a question from Markl [6].

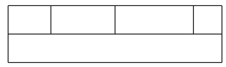
In this talk, we will mainly focus on sketching key components from (1) and (2) by harnessing a calculus of rectangles underlying box operads. In particular, we will present the following three results.

Box operads can be encoded as algebras over the symmetric coloured operad $\square\mathsf{p}$ (pronounced "boxop"). $\square\mathsf{p}$ consists of stackings of boxes which compose operadically by substituting a box in a stacking by a stacking of boxes. The following drawing provides an example of a stacking



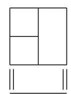
$\square$p generalizes the well-known symmetric coloured operad Op encoding operads, which is defined using trees [7].

On the other hand, box operads can equivalently be encoded as monoids in a category with a *skew* monoidal product $\square$, called the “box composite". We will introduce the box composite using two-level stackings, for example



Interestingly, skew monoidal categories recently also appeared in various other contexts, such as operadic categories [8].

Thirdly, to each box operad we are able to associate higher algebraic operations L_n constituting a L_∞-algebra. A crucial role is played by *thin* boxes: boxes whose vertical sides are degenerate. The operations L_n are obtained by summing over *thin-quadratic* stackings, i.e. stackings that are quadratic (in an appropriate sense) with respect to the thin boxes they contain. A thin-quadratic stacking that is not strictly quadratic is for example



Remark the bottom box is thin. This result generalizes the classical result for operads: quadratic trees induce a preLie-structure often called the Gerstenhaber brace.

Finally, I will explain briefly how these three ingredients play a key role in Koszul duality for box operads. If time permits, we will delve deeper by unpacking the (co)bar functor, twisting morphisms, the twisted complex and the application of Koszul duality to the box operad Lax.

Extending the above results to the coloured setting is an interesting topic for future research motivated by coloured versions of Lax. Part of this work is joint with Wendy Lowen and Hoang Dinh Van.

References

[1] T. Leinster, fc-multicategories, preprint arXiv:9903004, 1999.

[2] Cruttwell, G. & Shulman, M. A unified framework for generalized multicategories. *Theory Appl. Categ.*. **24** pp. No. 21, 580-655, 2010.

[3] Dinh Van, H., Hermans, L. & Lowen, W. Box operads and higher Gerstenhaber brackets. preprint arXiv:2305.20036, 2023.

[4] Dinh Van, H., Hermans, L. & Lowen, W. Operadic structure on the Gerstenhaber-Schack complex for prestacks. *Selecta Math. (N.S.)*. **28**, Paper No. 47, 63, 2022.

[5] Hermans, L. A minimal model for prestacks via Koszul duality for box operads. preprint arXiv:2312.04185, 2023.

[6] Markl, M. Homotopy diagrams of algebras. *Proceedings Of The 21st Winter School “Geometry And Physics” (Srni, 2001).*, 161-180, 2002.

[7] Laan, P. Coloured Koszul duality and strongly homotopy operads. preprint arXiv:0312147, 2004.

[8] Lack, S. Operadic categories and their skew monoidal categories of collections. *High. Struct.*. **2**, 1-29, 2018.

On predicate liftings and lax extensions of functors

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Abstract.
Lax extensions of Set-functors to bicategories of (enriched) relations are a well-established tool in various parts of mathematics: they are fundamental in our work on "monoidal topology" [4], but also generic notions of bisimulation for coalgebras rely on lax extensions [5, 6]. Furthermore, by providing the semantical framework to interpret modal operators, predicate liftings are at the heart of the standard approach to coalgebraic modal logic [1].

One of the principal motivation for our work is the paper [6] where, among others, it is shown that

- for a normal (= identity preserving) lax extension L of a functor $F\colon \mathsf{Set} \to \mathsf{Set}$, L-bisimilarity captures precisely behavioural equivalence of F-coalgebras,
- the double-powerset functor does not have a lax extension L so that "L-bisimilarity captures behavioural equivalence", therefore the double-powerset functor does not have a normal lax extension,
- a finitary functor $F\colon \mathsf{Set} \to \mathsf{Set}$ has a normal lax extension if and only if F has a separating set of monotone predicate liftings.

Having these results as starting point, in this talk we will

- give necessary and sufficient conditions, in terms of (weak) preservation of certain pullbacks, for a Set-functor to admit a normal lax extension to Rel, as well as a largest normal lax extension,
- discuss uniqueness of normal lax extensions to Rel,
- provide a point-free perspective on the connection between lax extensions and predicate liftings in the context of quantale-enriched relations. In particular, we introduce a notion of predicate lifting for a lax extension which leads to a simple description of Moss lifting that goes beyond the realm of accessible functors and is independent of functor presentations (which feature centrally in previous approaches), and we show that every quantale-valued lax extension of an arbitrary Set-functor is induced by its class of Moss liftings [2]. We note here that this result explains the importance of the canonical extensions of generalized monotone neighborhood functors in the process of constructing quantale-valued lax extensions (in analogy to the two-valued case [6]); it is a stepping stone to connecting the coalgebraic approaches to behavioural distance via quantale-valued lax extensions and via liftings to categories of quantale-enriched categories, respectively [3].

This talk is based on joint work with Pedro Nora (University of Nijmegen), Sergey Goncharov, Lutz Schröder and Paul Wild (Friedrich-Alexander-Universität Erlangen-Nürnberg).

References

[1] C. Cîrstea, A. Kurz, D. Pattinson, L. Schröder, and Y. Venema, *Modal Logics are Coalgebraic*, The Computer Journal (2011), 54, 31–41.

[2] S. Goncharov, D. Hofmann, P. Nora, L. Schröder, and P. Wild, *A Point-free Perspective on Lax extensions and Predicate liftings*, Mathematical Structures in Computer Science (2024), 34, 1–30.

[3] S. Goncharov, D. Hofmann, P. Nora, L. Schröder, and P. Wild, *Kantorovich Functors and Characteristic Logics for Behavioural Distances*, in Foundations of Software Science and Computation Structures, O. Kupferman and P. Sobocinski, eds., vol. 13992 of Lecture Notes in Computer Science, Cham, 2023, Springer Nature Switzerland, pp. 46–67. 26th International Conference, FoSSaCS 2023, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2023, Paris, France, April 22–27, 2023, Proceedings.

[4] D. Hofmann, G. J. Seal, and W. Tholen, eds., *Monoidal Topology. A Categorical Approach to Order, Metric, and Topology*, vol. 153 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, July 2014. Authors: Maria Manuel Clementino, Eva Colebunders, Dirk Hofmann, Robert Lowen, Rory Lucyshyn-Wright, Gavin J. Seal and Walter Tholen.

[5] J. Hughes and B. Jacobs, *Simulations in coalgebra*, Theoretical Computer Science (2004), 327, 71–108.

[6] J. Marti and Y. Venema, *Lax extensions of coalgebra functors and their logic*, Journal of Computer and System Sciences (2015), 81, 880–900.

Quotient toposes of discrete dynamical systems

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Abstract.

Lawvere's open problem on quotient toposes [1] has been solved for boolean Grothendieck toposes but not for non-boolean toposes. As a simple and non-trivial example of a non-boolean topos, we provide a complete classification of the quotient toposes of the topos of *discrete dynamical systems*. In this context, a discrete dynamical system means a pair (X, f) of a set X and an endofunction $f\colon X \to X$.

More concretely, our main theorem classifies the full subcategories of the topos $\mathbf{PSh}(\mathbb{N})$ that are closed under finite limits and small colimits. There are numerous such full subcategories, including those for which:

- Every state is in a loop.
$$\forall x \in X,\ \exists n > 0,\ f^n(x) = x.$$
- Every state is eventually fixed.
$$\forall x \in X,\ \exists n > 0,\ f^{n+1}(x) = f^n(x).$$
- f is bijective.
- Every state enters a loop within two steps, where the period of the loop has no square factors.
$$\forall x \in X,\ \exists n > 0,\ (f^{n+2}(x) = f^2(x)) \wedge (\forall p : \text{prime } p^2 \nmid n)$$

The goal of this talk is to describe these classes uniformly and clarify the background mathematical structures.

Our result is deeply related to monoid epimorphisms. Utilizing the theory of lax epimorphisms in the 2-category $\mathbf{Cat}$, we will explain how (non-surjective) monoid epimorphisms from $\mathbb{N}$ correspond to (non-periodic) behaviors in discrete dynamical systems.

This talk is based on the joint work with Yuhi Kamio [2].

References

[1] W. Lawvere, *Open problems in topos theory*, 2009. (One can find it via nLab. `https://ncatlab.org/nlab/show/William+Lawvere`)

[2] R. Hora, and Y. Kamio *Quotient toposes of discrete dynamical systems*, Journal of Pure and Applied Algebra, accepted 2024, in press.

The Elementary Theory of the 2-Category of Small Categories

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Abstract.

Lawvere's *Elementary Theory of the Category of Sets* (ETCS) [3] posits that the category **Set** is a well-pointed elementary topos with natural numbers object satisfying the axiom of choice. This provides a category theoretic foundation for mathematics which axiomatises the properties of function composition in contrast to Zermelo-Fraenkel set theory with the axiom of choice (ZFC), which axiomatises sets and their membership relation. Furthermore, ETCS augmented with the axiom schema of replacement can be shown to be equiconsistent with ZFC.

In this talk, I will present a categorification of ETCS which axiomatises the 2-category of small categories, functors and natural transformations; this is *the elementary theory of the* 2*-category of small categories* (ET2CSC) of the title. This extends Bourke's [1] characterisation of categories internal to a category $\mathcal{E}$ with pullbacks to the setting where $\mathcal{E}$ satisfies the extra properties of ETCS. Important 2-categorical definitions I will introduce are 2-well-pointedness, the full subobject classifier and the categorified axiom of choice. The main conclusion is that ET2CSC is 'Morita biequivalent' with ETCS, meaning that the two theories have biequivalent 2-categories of models. The proof of this uses various adjunctions between $\mathcal{E}$ and $\mathbf{Cat}(\mathcal{E})$ in order to transfer properties from one to the other.

I will also describe how Shulman and Weber's ideas on discrete opfibration classifiers can be used to incorporate replacement, in a way reminiscent of algebraic set theory.

This talk is based on joint work with Adrian Miranda [2].

References

[1] John Bourke. *Codescent objects in 2-dimensional universal algebra.* PhD thesis, University of Sydney, 2010.

[2] C. Hughes, A. Miranda, *The Elementary Theory of the* 2*-Category of Small Categories*, preprint arXiv:2403.03647, 2024.

[3] F William Lawvere. An elementary theory of the category of sets. *Proceedings of the National Academy of Sciences*, 52(6):1506–1511, 1964.

[4] M. Shulman, Classifying Discrete Opfibration page on ncatlab. https://ncatlab.org/michaelshulman/show/classifying+discrete+opfibration, (Accessed Nov 2023, Revision 15).

Automata in W-Toposes, and General Myhill-Nerode Theorems

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Abstract.

We extend the functorial approach to automata by Colcombet and Petrişan [1] from the category of sets to any W-topos and establish general Myhill-Nerode theorems in our setting, including an explicit relationship between the syntactic monoid and the transition monoid of the minimal automaton. As a special case we recover the result of Bojańczyk, Klin and Lasota [2] for orbit-finite nominal automata by considering automata in the Myhill-Schanuel topos of nominal sets.

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At the moment of the submission of this abstract (March 20), an article was conditionally accepted for CMCS 2024. A preprint can be found at Iwaniack [3].

References

[1] Thomas Colcombet and Daniela Petrişan. Automata Minimization: A Functorial Approach. *Logical Methods in Computer Science ; Volume 16*, page 1712.07121, March 2020.

[2] Mikołaj Bojańczyk, Bartek Klin, and Sławomir Lasota. Automata theory in nominal sets. *Logical Methods in Computer Science*, 10 issue 3(4):1402.0897, August 2014.

[3] Victor Iwaniack. Automata in toposes, and general Myhill-Nerode theorems, preprint arXiv:2307.14855, July 2023.

An Enriched Small Object Argument Over a Cofibrantly Generated Base

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Abstract.

The small object argument is a transfinite construction of weak factorization systems developed by Quillen [1], originally motivated by homotopy theory. Since then, various variations ([2, 1.37], [3], [4, 13.2.1]) of the small object argument became an important tool in category theory itself and also in other fields of mathematics such as model theory due to the connection between the argument and ubiquitous notions of injectivity and orthogonality.

In the talk I will tell you about an enriched variant of the small object argument that subsumes the classical 1-categorical small object argument for weak factorization systems, the 1-categorical small object argument for orthogonal factorization systems, and certain variants of the small object argument for 2-categories, (2,1)-categories, dg-categories and simplicially enriched categories.

Along the way, we will introduce a variation of the Day convolution in which we use copowers instead of the monoidal product. In more detail: Given a cosmos $\mathcal{V}$ and a $\mathcal{V}$-category $\mathcal{K}$, we introduce an analogue $F * X\colon \mathcal{A} \to \mathcal{K}$ of the Day convolution for $\mathcal{V}$-functors $F\colon \mathcal{A} \to \mathcal{V}$, $X\colon \mathcal{A} \to \mathcal{K}$, in which we use copowers in $\mathcal{K}$ instead of the monoidal product in $\mathcal{V}$. This then makes the underlying category $[\mathcal{A}, \mathcal{K}]_0$ of the $\mathcal{V}$-category $[\mathcal{A}, \mathcal{K}]$ of $\mathcal{V}$-functors $\mathcal{A} \to \mathcal{K}$ a copowered $[\mathcal{A}, \mathcal{V}]$-category. These copowers play a central role in our variant of the small object argument.

The talk will be based on the preprint [5].

References

[1] D.G. Quillen, *Homotopical Algebra*, Lecture Notes in Mathematics, No. 43 Springer-Verlag, Berlin-New York, 1967.

[2] J. Adámek, J. Rosický, *Locally presentable and accessible categories*, London Mathematical Society Lecture Note Series, 189. Cambridge University Press, Cambridge, 1994.

[3] R. Garner, *Understanding the small object argument*, Appl. Categ. Structures 17 (2009), no. 3, 247–285.

[4] E. Riehl, *Categorical homotopy theory*, New Mathematical Monographs, 24. Cambridge University Press, Cambridge, 2014.

[5] J. Jurka, An Enriched Small Object Argument Over a Cofibrantly Generated Base, preprint arXiv:2401.05974, 2024.

Formalizing accessibility and duality in a virtual equipment

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Abstract. Gabriel–Ulmer duality is a duality between logical theories and categories of their models. The categories of models are called locally presentable categories and are known to be characterized as ind-completions of theories. Accessible categories are a generalization of locally presentable categories and are also characterized as ind-completions. In the accessible context, there are many known "Gabriel–Ulmer-like" dualities, for example, Makkai–Paré duality [4], Adamek–Lawvere–Rosický duality [1], and so on. Even in the enriched accessible context, there are many kinds of duality [5]. Then, an axiomatic approach to these dualities is suggested by [3]. That approach works in a 2-category and characterizes ind-completions, which are the core concept of dualities, as a KZ-monad.

In this talk, we will give another axiomatic approach to accessibility and duality in a double-categorical setting. More precisely, inspired by [2], we will work in an (augmented) virtual equipment rather than a 2-category. We characterize an ind-completion as a vertical morphism having a "Yoneda-like" universal property, which we call an *ind-morphism*. Then, we will show that the ind-morphisms form a double-categorical counterpart of relative (bi)adjunctions and that it yields a duality theorem. This talk is based on joint work with Keisuke Hoshino.

References

[1] J. Adámek, F. W. Lawvere, J. Rosický, *On the duality between varieties and algebraic theories*, Algebra Universalis 49, (2003), no. 1, 35-49.

[2] S. R. Koudenburg, Formal category theory in augmented virtual double categories, preprint arXiv:2205.04890, 2022.

[3] I. Di Liberti, F. Loregian, *Accessibility and presentability in 2-categories*, J. Pure Appl. Algebra 227, (2023), no. 1, 107155.

[4] M. Makkai, R. Paré, *Accessible categories: the foundations of categorical model theory*, Contemporary Mathematics., vol. 104, American Mathematical Society, 1989.

[5] G. Tendas, Dualities in the theory of accessible categories, preprint arXiv:2302.06273, 2023.

On eigen-ring construction for monads

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Abstract.

For a ring A and its left ideal J, the eigen-ring [1] is defined by the quotient of the idealizer by J where the idealizer is the maximal subring of A which contains J as a two-sided ideal. Let k be a commutative unital ring. In this talk, we give a generalization of this concept by replacing rings with monads in the bicategory B_k whose objects are sets, morphisms are bi-indexed k-modules and 2-morphisms are intertwiners. This provides a uniform framework to understand some representations of categories which we explain below. As a fundamental result, for a monad $\mathtt{T}$, a left ideal $\mathtt{J} \subset \mathtt{T}$ and its eigen-ring $\mathrm{E}_{\mathtt{T}}(\mathtt{J})$, we give an adjunction between the category of $\mathtt{T}$-modules and the category of $\mathrm{E}_{\mathtt{T}}(\mathtt{J})$-modules. This adjunction is a generalization of the Morita equivalence between k-modules and modules over the matrix algebra.

Monads in B_k are equivalent with k-linear categories. Let $\mathtt{A}_{\mathcal{C}}$ be the monad corresponding to the k-linearization $k\mathcal{C}$ of a category $\mathcal{C}$. The purpose of this talk is to give specific left ideals which encode some properties of $\mathtt{A}_{\mathcal{C}}$-modules: to be precise, the category of $\mathtt{J}$-generated $\mathtt{A}_{\mathcal{C}}$-modules, which should be explained in this talk, is equivalent to the category of $\mathtt{A}_{\mathcal{C}}$-modules subject to that property. For example, if $\mathcal{C}$ is the opposite category gr° of finitely generated free groups, then the properties such as analyticity, polynomiality [3, 4, 6], outer property [5] and primitivity [2] correspond to certain left ideals $\mathtt{I}^{\nu}, \mathtt{I}^{d+1}, \mathtt{I}^{\mathsf{out}}, \mathtt{I}^{\mathsf{pr}}$ respectively. As one of our main results, the table below computes their eigen-rings where $\mathtt{P}^d$ is the monad related with augmentation ideals; $\mathtt{D}_{\mathfrak{Lie}}$ is the monad associated with Lie operad; $\mathtt{H}_0$ is the monad induced by the 0-th group homology of free groups. Moreover, the application of the above adjunction to each case leads to well-known adjunctions in the literature. In particular, the case of primitivity reproduces the universal enveloping algebra construction (more generally, Powell's construction [6]). This work is now in progress.

Monad $\mathtt{T}$	Property	Left ideal $\mathtt{J}$	Eigen-monad $\mathrm{E}_{\mathtt{T}}(\mathtt{J})$
$\mathtt{A}_{\mathsf{gr}^{\circ}}$	polynomial with degree $\leq d$	$\mathtt{I}^{d+1}$	$\mathtt{P}^0/\mathtt{P}^{d+1}$
	analytic	$\mathtt{I}^{\nu}$	$\mathrm{E}_{\mathtt{A}_{\mathcal{C}}}(\mathtt{I}^{\nu}_{\mathcal{C}})$
	primitive	$\mathtt{I}^{\mathsf{pr}}$	$\mathtt{D}_{\mathfrak{Lie}}$
	outer	$\mathtt{I}^{\mathsf{out}}$	$\mathtt{H}_0$

References

[1] Ore, Oystein. "Formale Theorie der linearen Differentialgleichungen.(Zweiter Teil)." (1932): 233-252.

[2] Kim, Minkyu, and Christine Vespa. "On analytic exponential functors on free groups." arXiv preprint arXiv:2401.09151 (2024).

[3] Eilenberg, Samuel, and Saunders MacLane. "On the groups $H(\Pi, n)$, II: methods of computation." Annals of Mathematics (1954): 49-139.

[4] Hartl, Manfred, Teimuraz Pirashvili, and Christine Vespa. "Polynomial functors from algebras over a set-operad and nonlinear Mackey functors." International Mathematics Research Notices 2015.6 (2015): 1461-1554.

[5] Powell, Geoffrey, and Christine Vespa. "Higher Hochschild homology and exponential functors." arXiv preprint arXiv:1802.07574 (2018).

[6] Powell, Geoffrey. "On analytic contravariant functors on free groups." arXiv preprint arXiv:2110.01934 (2021).

Limits in Enhanced Simplical Categories

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Abstract.

In [3], Riehl and Verity have developed the theory of ∞-*cosmoi*, which are quasi-categorically enriched categories that satisfy certain nice properties resembling enriched categories of fibrant objects. The theory of ∞-cosmoi provides a setting for understanding models for $(\infty, 1)$-categories with structures, and the pseudo morphisms between them. For instance, we have the ∞-cosmos of $(\infty, 1)$-categories admitting limits of shape J for a simplicial set J and the functors which preserve limits, and also the ∞-cosmos of Cartesian fibrations between $(\infty, 1)$-categories and the Cartesian functors.

Riehl and Verity have established in [3] that ∞-cosmoi admit all *flexible weighted limits*, which are simplicially enriched limits that are analogous to *PIE limits* in 2-category theory. Examples include products, inserters, and comma objects.

Besides, in [1], Lack has shown that the existence of 2-dimensional limits involving *lax* morphisms is subtle. For instance, in the 2-category of categories with limits of shape J and the functors that do *not* necessarily preserve limits, comma objects exist only when one of the 1-morphisms in the diagram preserves limits.

This phenomenon led Lack and Shulman to introduce *enhanced* 2-*category theory* in [2]. An enhanced 2-category is a 2-category with *two* types of 1-morphisms: the tight ones and the loose ones, in which every tight 1-morphism is also loose. For example, categories admitting J-shaped limits form an enhanced 2-category, with the tight 1-morphisms given by the functors that preserve limits, whereas the loose 1-morphisms given by just the functors, and the 2-morphisms given by the natural transformations. As enhanced 2-categories can be seen as enriched categories, Lack and Shulman have studied the subtle phenomenon of the existence of 2-dimensional limits involving *lax* morphisms via enriched category theory.

Taking inspiration from the work [2] by Lack and Shulman, we introduce the notion of *enhanced simplicial categories*, which are basically simplicial categories with *two* types of 0-arrows. Similarly, an enhanced simplicial category can be seen as an enriched category, hence we apply enriched category theory to study limits for *lax* morphisms in the ∞-categorical setting.

In the talk, we show that many interesting enhanced simplicial categories, such as that of $(\infty, 1)$-categories possessing limits together with the pseudo and lax morphisms between them, admit certain weighted limits. These include comma objects with one tight 0-arrow in the diagram, ∞-categorical versions of equifiers and inserters, and some further advanced limits, all involving loose 0-arrows in the corresponding diagrams. In particular, these results specialise to any model for $(\infty, 1)$-categories, generalising results on quasi-categories and also categories.

References

[1] S. Lack, *Limits for lax morphisms*, Appl. Categ. Structures 13 (2005), no. 3, 189–203.

[2] S. Lack, and M. Shulman, *Enhanced 2-categories and limits for lax morphismss*, Adv. Math. 229 (2012), no. 1, 294–356.

[3] E. Riehl, and D. Verity, *Elements of* ∞*-Category Theory*, first ed., Cambridge Studies in Advanced Mathematics, vol. 194, Cambridge University Press, 2022.

The universal property of the decomposition space of quasisymmetric functions

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Abstract. The coalgebra **QSym** of quasisymmetric functions was shown by Aguiar, Bergeron, and Sottile [1] to be the terminal object in the category of graded coalgebras with a zeta function. I'll explain an objective version of that result: **QSym** is the incidence coalgebra of a decomposition space Q of monotone surjections, and its zeta function Z is given by the empty surjection and the connected surjections [2]. We show that for any graded decomposition space X with a zeta function F, there is a unique graded span of decomposition spaces

$$X \xleftarrow{\alpha} J \xrightarrow{\varphi} Q$$

where α is ikeo (inner Kan and equivalence on objects) and φ is culf (conservative and unique lifting of factorisations) inducing F from Z. (Such spans induce coalgebra homomorphisms, and conjecturally all.)

This is joint work with Philip Hackney and Jan Steinebrunner [3].

References

[1] Marcelo Aguiar, Nantel Bergeron, and Frank Sottile. *Combinatorial Hopf algebras and generalized Dehn-Sommerville relations.* Compos. Math. **142** (2006), 1–30.

[2] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. *Decomposition spaces of quasi-symmetric functions.* Unpublished.

[3] Philip Hackney, Joachim Kock, and Jan Steinebrunner. *The universal property of the decomposition space of quasisymmetric functions.* In preparation.

Formal Day convolution and low-dimensional monoidal fibrations

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Abstract.

Let T be a monad on an augmented virtual double category $\mathcal{K}$, the latter in the sense of [1]. The main result of this talk describes conditions ensuring that a formal Yoneda embedding $y\colon A \to P$ in $\mathcal{K}$ (in the sense of [2]) can be lifted along the forgetful functor $U\colon \mathsf{Lax}{-}T{-}\mathsf{Alg} \to \mathcal{K}$, where $\mathsf{Lax}{-}T{-}\mathsf{Alg}$ is the augmented virtual double category of lax T-algebras.

Taking $\mathcal{K} = \mathsf{Prof}$ the augmented virtual double category of profunctors and T the "free strict monoidal category"-monad the main result recovers the Day convolution monoidal structure on the category of presheaves $P = \mathsf{Set}^{A^{\mathrm{op}}}$ on a monoidal category A. Taking the same monad on the augmented virtual double category $\mathcal{K} = \mathsf{dFib}$ of two-sided discrete fibrations instead, the main result implies the "monoidal Grothendieck equivalence" of lax monoidal functors $A \to \mathsf{Set}$ and monoidal discrete opfibrations with base A (a variation on a result in [3] by Moeller and Vasilakopoulou).

Moving up a dimension, given a 2-monoidal 2-category A the main result likewise implies the equivalence of lax 2-monoidal 2-functors $A \to \mathsf{Cat}$ and 2-monoidal locally discrete split 2-opfibrations with base A. The main ingredient here is that (somewhat surprisingly) there exists an augmented virtual double category that accommodates the lax natural transformations required to define the formal Yoneda embedding induced by the Grothendieck equivalence for locally discrete split 2-opfibrations (the latter obtained by Buckley [4] and Lambert [5]).

I will report on work in progress on "internalising" the equivalence for 2-monoidal locally discrete split 2-opfibrations described above, thus obtaining an analogous equivalence for monoidal double split opfibrations (double fibrations in the sense of Cruttwell, Lambert, Pronk and Szyld [6]).

References

[1] S. R. Koudenburg. Augmented virtual double categories. Theory and Applications of Categories, 35(10):261–325, 2020.

[2] S. R. Koudenburg. Formal category theory in augmented virtual double categories. Preprint, arXiv:2205.04890, 2022.

[3] J. Moeller and C. Vasilakopoulou. Monoidal Grothendieck construction. Theory and Applications of Categories, 35(31):1159–1207, 2020.

[4] M. Buckley. Fibred 2-categories and bicategories. Journal of Pure and Applied Algebra, 218(6):1034–1074, 2014.

[5] M. Lambert. Discrete 2-Fibrations. Preprint, arXiv:2001.11477, 2020.

[6] G. S. H. Cruttwell, M. J. Lambert, and D. A. Pronk and M. Szyld. Double fibrations. Theory and Applications of Categories, 38(35):1326–1394, 2022.

Poc sets and median algebras: A categorical duality

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Abstract.

The construction of a monad from an adjunction has a generalization allowing certain kinds of functors to induce monads as well. In particular, a functor whose target is rich enough in limits induces a codensity monad. In the presence of a left adjoint, the codensity monad of a functor agrees with the corresponding adjunction-induced monad. Since inclusion functors associated to subcategories of finite objects are unlikely to admit a left adjoint, their codensity monads are of interest. Familiar categories to consider are sets or vector spaces. In these cases, we get the ultrafilter and double dualization monads [1]. The goal of this talk is to compute a dual pair of codensity monads analogous to those just mentioned.

An important topic in geometric group theory is the study of group actions on trees. For this purpose, two vast generalizations of trees arose independently: poc sets and median algebras. Boolean algebras are an example of both, and in fact any set that simultaneously has suitably compatible structures of a poc set and median algebra is a Boolean algebra. The two-element Boolean algebra represents a pair of dualization functors between the categories of poc sets and median algebras. The dual of a poc set is its median algebra of ultrafilters, and the dual of a median algebra is its poc set of halfspaces [2]. In this talk, I will demonstrate that this pair of dualization functors exhibits similar properties to the dualization functor on vector spaces. I will show how dualization fits into a two-variable tensor-hom adjunction, and that both double dualization monads are the codensity monads of inclusions from finite objects of each category. Time permitting, I will say more about the kinds of Boolean algebras one can construct as the tensor product of a poc set and a median algebra.

References

[1] T. Leinster, *Codensity and the ultrafilter monad*, Theory Appl. Categ. 28 (2013), No. 13, 332-370

[2] M.A. Roller,*Poc sets, median algebras and group actions. An extended study of Dunwoody's construction and Sageev's theorem*, Preprint, Univ. of Southampton, 1998.

Higher Groupoids and Higher Generalised Morphisms

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Abstract.
Higher groupoids play a crucial role in the active research area of interplay between higher categorical structures and other fields of mathematics. We give the notion of a Good Geometric category, where one can define and study these higher structures with applications to geometry, for example category of smooth manifolds. We define the notion of higher groupoids in Good Geometric categories and organise them into an $(\infty, 1)$ category framework. The morphisms between the higher groupoids are given by bibundles which are Kan fibrations over the interval. Higher morphisms will be modelled by Kan fibrations over the higher simplices. This approach gives a more combinatorial and geometric way of approaching anafunctors and higher generalised morphisms between groupoids. This is of particular interest in higher gauge theory and string theory, where the higher connection on higher bundles will give the notion of parallel transport of strings and surfaces.

References

[1] C. Blohmann, K. Krishna, and C. Zhu, *Higher Generalised Morphisms and Morita Equivalence for Higher Groupoids*, Work in Progress.

[2] D. Li, *Higher Groupoid Actions, Bibundles, and Differentiation*, PhD thesis, University of Göttingen, 2015.

[3] A. Joyal, *Quasi-categories and Kan complexes*, Journal of Pure and Applied Algebra, 175(1):207-222, 2002.

[4] J. Lurie, *Higher Topos Theory*, AM(170), Princeton University Press, 2009.

The Grothendieck construction in the context of tangent categories

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Abstract.
Cockett and Cruttwell in their investigation of vector bundles in the context of a tangent category (cf. [3]), which led to the concept of differential bundles, came out with an important generalization of the notion of fibrations: tangent fibrations. In a nutshell, a tangent fibration is a fibration between two tangent categories, which is also a strict tangent morphism. Cockett and Cruttwell also realized that, by pulling back along the zero morphism of the base tangent structure, the fibres of a tangent fibration inherit a tangent structure. So, a tangent fibration can be sent to an indexed tangent category, which is an indexed category whose fibres have a tangent structure, in a compatible way with the indexing and the substitution functors between them.

Famously, the Grothendieck construction establishes an equivalence between (cloven) fibrations and indexed categories (cf. [2]), so it is natural to wonder whether or not the operation introduced by Cockett and Cruttwell which sends a tangent fibration to its indexed tangent category can be extended to an equivalence. In particular, the question is whether or not we can reconstruct the total tangent structure, i.e. the tangent structure over the total category, of the tangent fibration, starting from the associated indexed tangent category.

The answer is only partial: by pulling back via the zero morphism we inevitably lose some information about the total tangent structure. To solve this issue I explored a generalization of tangent structures: the notion of tangent objects. A tangent object consists of an object in a given 2-category together with a tangent structure on it. It is important to mention that in his thesis, Leung went close to introducing this concept, by generalizing tangent structures over an arbitrary monoidal category (see [4]. In my version, I generalize this notion over a 2-category instead) and Bauer, Burke, and Ching employed a similar idea to introduce tangent ∞-categories in [1].

This simple generalization of a tangent category leads to interesting questions and new approaches to the theory of tangent categories. In particular, tangent objects in the 2-category of fibrations over a non-fixed base category are precisely tangent fibrations. From this observation, I proved the main result: tangent fibrations are equivalent to tangent indexed categories (to not be confused with indexed tangent categories mentioned earlier), which are tangent objects in the 2-category of indexed categories over a non-fixed indexing category.

In my talk, I would like to briefly recall the definition of a fibration, of an indexed category, and the Grothendieck construction in the classical case. Then, I would like to introduce the notion of tangent fibrations, as presented by Cockett and Cruttwell, and briefly discuss their result which leads to the notion of indexed tangent categories. I plan to discuss how to partially reconstruct the original tangent fibration from the associated indexed tangent category and show what fails to be recovered.

I will dedicate some time to introduce the main new technology: tangent objects, discuss a few important examples, like tangent categories, which are tangent objects in the 2-category of categories, tangent monads, which turn out to be tangent objects in the 2-category of monads, and finally tangent fibrations, as tangent objects in the 2-category of fibrations. Finally, I would like to unpack the definition-theorem of the equivalence between tangent fibrations and the new notion of tangent indexed categories.

My paper is available on Arxiv here:
https://arxiv.org/abs/2311.14643

References

[1] K. Bauer, M. Burke, and M. Ching, Tangent infinity-categories and Goodwillie calculus, preprint https://arxiv.org/abs/2101.07819, 2023.

[2] F. Borceux, *Handbook of Categorical Algebra 2 – Categories and Structures*, Cambridge Univ. Press (1994).

[3] R. Cockett, G. Cruttwell, *Differential Bundles and Fibrations for Tangent Categories*, Cahiers de Topologie et Géométrie Différentielle Catégoriques (2018), vol. LIX, 10–92.

[4] P. Leung, *Tangent bundles, monoidal theories and Weil algebras*, Bulletin of the Australian Mathematical Society, vol. 98 no. 1, 2018.

Identity objects and virtualisation

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Abstract.
In categorical logic, indexed preorders are an interpretation of many-sorted predicate logic. Taking the view that many-sorted predicate logic is a highly truncated version of dependent type theory, we obtain the following adaptation of the inductive axioms of identity types [1] to indexed preorders.

Definition. Let $P^1 \colon (P^0)^{\mathrm{op}} \to \mathrm{Pre}^{\wedge,\top}$ be an indexed $(\wedge,\top)$-preorder over a binary-product category P^0. An *identity object* on an object $X \in P^0$ is an element $\mathrm{Id}_X \in P^1(X \times X)$, such that

1. (*introduction* or *reflexivity*) $\top \leq (X \xrightarrow{\delta} X \times X)^*(\mathrm{Id}_X)$, and
2. (*elimination*) for any object $Y \in P^0$ and $p, q \in P^1(X \times X \times Y)$, if
$$(X \times Y \xrightarrow{\delta \times Y} X \times X \times Y)^*(p) \leq (X \times Y \xrightarrow{\delta \times Y} X \times X \times Y)^*(q),$$
then $(X \times X \times Y \xrightarrow{\pi_1;\pi_2} X \times X)^*(\mathrm{Id}_X) \wedge p \leq q$.

We say $P \coloneqq (P^0, P^1)$ *has identity objects* if each X has an identity object.

This Martin-Löf notion of equality turns out, perhaps as expected, to be equivalent to Lawvere's one as extracted by Maietti and Rosolini in the notion of elementary doctrine [2]:

Theorem. *An indexed $(\wedge,\top)$-poset over a finite-product category has identity objects if and only if it is an elementary doctrine.*

This means Pasquali's 'elementary completion' result [4] is telling us that the *equivalence relations construction* $P \mapsto \mathrm{ER}(P)$ underlying Maietti and Rosolini's 'effective-quotient completion' [3] is a right-biadjoint completion that adds identity objects. Pasquali's result adapted to our settings reads:

Theorem. *The assignment $P \mapsto \mathrm{ER}(P)$ extends to a 2-functor $\mathrm{IdxPre}^{\times,\wedge,\top}_{\mathrm{pn}} \to \mathrm{IdxPre}^{\times,\wedge,\top,\mathrm{Id}}_{\mathrm{pn}}$ that is right biadjoint to the inclusion 2-functor.*

Here, the notation e.g. $\mathrm{IdxPre}^{\times,\wedge,\top,\mathrm{Id}}_{\mathrm{pn}}$ denotes the 2-category of indexed $(\wedge,\top)$-preorders with identity objects over binary-product categories, **p**seudo**n**atural morphisms that preserves $\times$, $\wedge$, $\top$ and Id, and 2-morphisms; these morphisms and 2-morphisms are defined in the same way as in [2, 3, 4], except that our morphisms have a *pseudo*natural-transformation component.

We produce an analogue of this result for the *PER construction*, the partial equivalence relations version of the ER construction, which appears as a key step in the tripos-to-topos construction [5].

Let P be an indexed $\wedge$-preorder over a binary-product category. The PER construction is given by an indexed preorder $\mathrm{PER}(P)$ defined in the same way as $\mathrm{ER}(P)$ but with as objects in $\mathrm{PER}(P)^0$ *partial* equivalence relations in P instead. Now the following weakened form of identity objects allows us to render the PER construction as a right-biadjoint completion.

Definition. We say P *has partial identity objects* if each object $X \in P^0$ is *equipped* with an element $\mathrm{PId}_X \in P^1(X \times X)$, such that

1. (*partial reflexivity*) $\mathrm{PId}_X \leq (X \times X \overset{\pi_1}{\to} X \times X)^*(\mathrm{PId}_X), (X \times X \overset{\pi_2}{\to} X \times X)^*(\mathrm{PId}_X)$,
2. (*paravirtual elimination*) for each object $Y \in P^0$ and elements $p, q \in P^1(X \times X \times Y)$, if

$$(X \times Y \overset{\pi_1;\pi_1}{\to} X \times X)^*(\mathrm{PId}_X) \wedge (X \times Y \overset{\pi_2;\pi_2}{\to} Y \times Y)^*(\mathrm{PId}_Y) \wedge$$
$$(X \times Y \overset{\delta \times Y}{\to} X \times X \times Y)^*(p) \leq (X \times Y \overset{\delta \times Y}{\to} X \times X \times Y)^*(q),$$

   then $(X \times X \times Y \overset{\pi_3;\pi_3}{\to} Y \times Y)^*(\mathrm{PId}_Y) \wedge (X \times X \times Y \overset{\pi_1;\pi_2}{\to} X \times X)^*(\mathrm{PId}_X) \wedge p \leq q$,
3. each arrow $f\colon X \to Y$ in P^0 satisfies $\mathrm{PId}_X \leq (f \times f)^*(\mathrm{PId}_Y)$, and
4. $\mathrm{PId}_{X \times Y} \simeq (X \times Y \times X \times Y \overset{\pi_1;\pi_3}{\to} X \times X)^*(\mathrm{PId}_X) \wedge (X \times Y \times X \times Y \overset{\pi_2;\pi_4}{\to} Y \times Y)^*(\mathrm{PId}_Y)$.

Theorem. *The assignment $P \mapsto \mathrm{PER}(P)$ extends to a 2-functor $\mathrm{IdxPre}^{\times,\wedge}_{\mathrm{pn}} \to \mathrm{IdxPre}^{\times,\wedge,\mathrm{PId}}_{\mathrm{pn}}$ that is right biadjoint to the forgetful 2-functor.*

Indexed preorders with partial identity objects can be promoted to those with identity objects by a construction we call *virtualisation*. This is in fact another step in the tripos-to-topos construction, which turns the PERs into equivalence relations. An indexed preorder P is *oplaxly sectioned* if each object $X \in P^0$ is equipped with an element $\mathrm{os}_X \in P^1(X)$, and every arrow $f\colon X \to Y$ in P^0 satisfies $\mathrm{os}_X \leq f^*(\mathrm{os}_Y)$. We regard an indexed preorder with partial identity objects as oplaxly sectioned, by $\mathrm{os}_X \coloneqq (X \overset{\delta}{\to} X \times X)^*(\mathrm{PId}_X)$. Let P be an oplaxly sectioned indexed $\wedge$-preorder.

Definition. The *virtualisation* of P is the indexed preorder $\mathrm{Virt}(P)$ given by $\mathrm{Virt}(P)^0 \coloneqq P^0$ and $\mathrm{Virt}(P)^1(X) \coloneqq (\mathrm{U}_{\mathrm{Set}} P^1(X), \overset{\mathrm{v}}{\leq})$ where $p \overset{\mathrm{v}}{\leq} q$ if and only if $\mathrm{os}_X \wedge p \leq q$.

$\mathrm{Virt}(P)^1$ is in fact a Kleisli as well as Eilenberg-Moore object for a (necessarily idempotent) comonad in the Pre-enriched category $[(P^0)^{\mathrm{op}}, \mathrm{Pre}^{\wedge}]_{\mathrm{o}}$ of functors and *oplax* natural transformations.

Note that the os_X become top elements in $\mathrm{Virt}(P)$, and if P has partial identity objects, then the PId_X become identity objects in $\mathrm{Virt}(P)$. Virtualisation has the following universal properties; beware that mainly **o**plax-**n**atural morphisms are involved here, rather than pseudonatural morphisms.

Theorem. *The assignment $P \mapsto \mathrm{Virt}(P)$ extends to a 2-functor $\mathrm{IdxPre}^{\wedge,\mathrm{os}}_{\mathrm{on}} \to \mathrm{IdxPre}^{\wedge,\top}_{\mathrm{on}}$ as well as a 2-functor $\mathrm{IdxPre}^{\times,\wedge,\mathrm{PId}}_{\mathrm{on}} \to \mathrm{IdxPre}^{\times,\wedge,\top,\mathrm{Id}}_{\mathrm{on}}$ that is ambidextrously biadjoint to 'the' respective inclusion 2-functor. The left-biadjoint part also holds with respect to pseudonatural morphisms.*

References

[1] P. Martin-Löf, *An intuitionistic theory of types: predicative part*, Logic Colloquium '73, Elsevier, 1975, pp. 73–118.

[2] M.E. Maietti and G. Rosolini, *Quotient Completion for the Foundation of Constructive Mathematics*. Log. Univers. 7 (2013), pp. 371–402.

[3] M.E. Maietti and G. Rosolini, *Elementary quotient completion*. TAC 27 (2013), no. 17, pp. 445–463.

[4] F. Pasquali, *A Co-free Construction for Elementary Doctrines*, App. Cat. Str. 23 (2015), pp. 29–41.

[5] J.M.E. Hyland, P.T. Johnstone and A.M. Pitts, *Tripos theory*. Math. Proc. Camb. Philos. Soc. 88 (1980), no. 2, pp. 205–232.

Free Differential Storage Modalities

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Abstract.

Storage modalities provide the categorical interpretation of the exponential modality from Linear Logic, while differential storage modalities [1, 2] do the same in Differential Linear Logic. Briefly, a storage modality on a symmetric monoidal category with finite products is a comonad ! such that every $!A$ is naturally a cocommutative comonoid and we have the Seely isomorphism $!(A \times B) \cong !A \otimes !B$. A differential storage modality on an additive symmetric monoidal category with finite biproducts is a storage modality ! which comes equipped with a natural transformation $\mathsf{d} : !A \otimes A \to !A$, called the deriving transformation, whose axioms are based on the fundamental identities of differentiation such as the product rule and the chain rule. Using Kelly's notion of algebraically-free commutative monoids [3], we construct free differential storage modalities over storage modalities. A symmetric monoidal category is said to be endowed with algebraically-free commutative monoids if for every object X, there is an object $\mathsf{S}(X)$ equipped with a map $\mathsf{S}(X) \otimes X \to \mathsf{S}(X)$ which is universal amongst commutative right X-actions $A \otimes X \to X$. Then for an additive symmetric monoidal category with finite biproducts which is endowed with algebraically-free commutative monoids, for every storage modality !, we get that $!(-) \otimes \mathsf{S}(-)$ is the free differential storage modality over !. In other words, in this setting, the forgetful functor from the category of differential storage modalities to the category of storage modalities has a left adjoint. Moreover, when taking ! to be the initial storage modality, we get the initial differential storage modality which is related to the Faà di Bruno construction [2] and also recaptures the exponential modality in Clift and Murfet's Differential Linear Logic model [4].

References

[1] Blute, R. F., Cockett, J. R. B., Lemay, J.-S. P., & Seely, R. A. (2020). *Differential categories revisited.* Applied Categorical Structures, 28, 171-235.

[2] Garner, R., & Lemay, J.-S. P. (2021). Cartesian differential categories as skew enriched categories. Applied Categorical Structures, 29(6), 1099-1150.

[3] Kelly, G. M. (1980). A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on. Bulletin of the Australian Mathematical Society, 22(1), 1-83.

[4] Clift, J., & Murfet, D. (2020). *Cofree coalgebras and differential linear logic.* Mathematical Structures in Computer Science, 30(4), 416-457.

Homotopy colimits enriched over a general base

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Abstract.

Starting from a 1-categorical base $\mathcal{V}$ which is *not* assumed endowed with a choice of model structure (or any kind of homotopical structure), we define homotopy colimits enriched in $\mathcal{V}$ in such a way that, for $\mathcal{V} = \mathsf{Set}$, we retrieve the classical theory as presented in [1] and [3]. We construct the free homotopy $\mathcal{V}$-cocompletion of a small $\mathcal{V}$-category and show that it satisfies the expected universal property. For $\mathcal{V} = \mathsf{Set}$, we retrieve Dugger's construction of the universal homotopy theory on a small category $\mathcal{C}$. We define the homotopy theory of internal ∞-groupoids in $\mathcal{V}$ as the homotopy $\mathcal{V}$-cocompletion of a point, and argue that $\mathcal{V}$-enriched homotopy colimits correspond to colimits in ∞-categories enriched in internal ∞-groupoids in $\mathcal{V}$, thus providing a convenient model to perform computations. Again, taking $\mathcal{V} = \mathsf{Set}$, this retrieves the classical notions for ordinary $(\infty, 1)$-categories. We compare our approach with some previous definitions of enriched homotopy colimits, such as those in [4] and [6].
As an application, we settle, for any group, a conjecture that in the case of a finite group was recently proven by completely different methods in [5]: we show that the so-called genuine homotopy theory of G-spaces is the G-equivariant homotopy cocompletion of a point. We conclude providing further examples of homotopy theories that can be seen as homotopy $\mathcal{V}$-cocompletions for a suitable choice of enrichment.

References

[1] W. Dwyer, P. S. Hirschhorn, D. Kan, J. Smith, *Homotopy Limit Functors on Model Categories and Homotopical Categories*, Mathematical Surveys and Monographs, Volume 113, American Mathematical Society, 2004.

[2] D. Dugger, *Universal Homotopy Theories*, Advances in Mathematics 164, 2001, pp. 144-176.

[3] P. S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, volume 99, American Mathematical Society, 2003.

[4] S. Lack and J. Rosický, *Homotopy locally presentable enriched categories*, Theory and Applications of Categories, Vol. 31, No. 25, 2016, pp. 712–754.

[5] J. Shah, *Parametrized higher category theory*, Algebraic and Geometric Topology 23, 2023, 509–644.

[6] M. Shulman, *Homotopy limits and colimits and enriched homotopy theory*, arXiv:math/0610194, 2009.

Logical Structure in (Homotopical) Inverse Functor Categories

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Abstract.

Reedy categories have been the subject of interest in categorical homotopy theory due to their ability to give an explicit, yet general, construction of the model structure on the category of simplicial objects in a model category. Inverse categories, which are the special case of Reedy categories where all (non-invertible) maps lower degree, have also been of interest: for instance, in type theory, [3] used them to provide a partial solution to the homotopy canonicity hypothesis, while [1] used them to put a model structure on the category of models of type theory. In both of these cases, one shows that if $\mathcal{E}$ is a model of type theory and $\mathcal{I}$ is an inverse category then $\mathcal{E}^{\mathcal{I}}$ forms a model of type theory as well. In [2], it is further shown that the category of homotopical diagrams $\mathcal{E}^{\mathcal{I}^{-1}\mathcal{I}}$ (that is, the full subcategory of functors $\mathcal{I} \to \mathcal{E}$ inverting all maps) is closed under much of the type-theoretic logical structure of $\mathcal{E}^{\mathcal{I}}$. While this result is sufficient for most type-theoretic considerations, it is natural to consider generalisations when $\mathcal{I}$ is equipped with weak equivalences $\mathcal{W}$. In this talk, we further consider the following questions and provide affirmative answers to them.

- Can one find conditions on $\mathcal{W}$ such that $\gamma^* \colon \mathcal{E}^{\mathcal{W}^{-1}\mathcal{I}} \to \mathcal{E}^{\mathcal{I}}$ is ensured to preserve the logical structure? In particular, as the techniques of [2] used to show the preservation of dependent products rely heavily on the assumption that all maps are inverted in $\mathcal{I}$, can one weaken this condition?
- Can one isolate the type-theoretic aspects from the logical aspects? For example, can one show that if $\mathcal{E}$ is a topos then so is $\mathcal{E}^{\mathcal{I}}$?

Here, we take advantage of the inverse structure of $\mathcal{I}$ to prove, for a category $\mathcal{E}$ (not necessarily a topos or a model of type theory), the following.

Definition. *For each $i \in \mathcal{I}$, write $\mathcal{I}^-(i)$ for the full subcategory of $i/\mathcal{I}$ spanned by the strictly degree-lowering maps, write $\partial(i/\mathcal{W}^{-1}\mathcal{I})$ for the full subcategory of $i/\mathcal{W}^{-1}\mathcal{I}$ with the initial object removed, and write $\mathbb{G}_n\mathcal{I}$ for the full subgroupoid of $\mathcal{I}$ spanned by the objects of degree $n \in \mathbb{N}$. Further, for each $i \in \mathcal{I}$, denote by $M_i \colon \mathcal{E}^{\mathcal{I}} \to \mathcal{E}$ the matching object functor.*

Theorem 1. *Assume that, for each $i \in \mathcal{I}$, the restriction of the localisation $\gamma \colon \mathcal{I} \to \mathcal{W}^{-1}\mathcal{I}$ given by $\gamma|_i \colon \mathcal{I}^-(i) \to \partial(i/\mathcal{W}^{-1}\mathcal{I})$ is an initial functor and that all limits indexed by $\mathcal{I}^-(i)$ exist in $\mathcal{E}$. Then, if dependent products along $f \colon B \to A$ in $\mathcal{E}^{\mathcal{W}^{-1}\mathcal{I}}$ exist, one also has dependent products along $\gamma^* f \colon \gamma^* B \to \gamma^* A$ in $\mathcal{E}^{\mathcal{I}}$. Furthermore, the canonical map $\gamma^* \Pi_f \Rightarrow \Pi_{\gamma^* f}\, \gamma^* \colon \mathcal{E}^{\mathcal{W}^{-1}\mathcal{I}}/B \to \mathcal{E}^{\mathcal{I}}/\gamma^* A$ is an isomorphism.*

This result generalises that of [2]; for instance, when $\mathcal{I} = \cdot \to \cdot \to \cdot$, inverting either of the arrows preserves dependent products.

In proving Theorem 1, we show the following.

Lemma 2. *For a functor $F \colon \mathcal{D} \to \mathcal{C}$ that preserves pullbacks, the dependent product in $\mathcal{C} \downarrow F$ along $(f, g) \colon (b, y, \beta) \to (a, x, \alpha)$*

$$\begin{tikzcd} b \arrow[r, dashed, "f"] \arrow[d, "\beta"'] & a \arrow[d, "\alpha"] \\ Fy \arrow[r, dashed, "Fg"'] & Fx \end{tikzcd}$$

exists provided that dependent products along $f \colon b \to a$ and $Fg \colon Fy \to Fx$ in $\mathcal{C}$, and $g \colon y \to x$ in $\mathcal{D}$ exist.

One concludes Theorem 1 using that Reedy induction in the inverse case, as also noted by [4], allows the construction of indexed diagrams by iterative gluing. Also by way of iterative gluing, we further show the following.

Theorem 3. *Assume all limits indexed by $\mathcal{I}^-(i)$ exist in $\mathcal{E}$ for each $i \in \mathcal{I}$. If $\mathcal{E}$ has a subobject classifier, then so does $\mathcal{E}^{\mathcal{I}}$ provided that either $\mathbb{G}_n\mathcal{I}$ is connected for each $n \in \mathbb{N}$ or $\mathcal{E}$ has an initial object. In addition, and independent of subobject classifiers, dependent products along $f \colon B \to A$ in $\mathcal{E}^{\mathcal{I}}$ exist provided that dependent products along $f_i \colon B_i \to A_i$ and $M_i f \colon M_i B \to M_i A$ in $\mathcal{E}$ exist for each $i \in \mathcal{I}$.*

Our result, in contrast to the type-theoretic results of [2], centres on the logical structure of categories of diagrams; for instance, when $\mathcal{I}$ is an inverse category and $\mathcal{E}$ is a topos then so is $\mathcal{E}^{\mathcal{I}}$.

References

[1] Krzysztof Kapulkin and Peter LeFanu Lumsdaine. The homotopy theory of type theories. *Advances in Mathematics*, 337:1–38, 2018.

[2] Krzysztof Kapulkin and Peter LeFanu Lumsdaine. Homotopical inverse diagrams in categories with attributes. *Journal of Pure and Applied Algebra*, 225(4):1–44, 2021.

[3] Michael Shulman. Univalence for inverse diagrams and homotopy canonicity. *Mathematical Structures in Computer Science*, 25(5):1203–1277, November 2014.

[4] Michael Shulman. Reedy categories and their generalizations, 2015.

Semicartesian categories of relations

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Abstract.

Quantization is the process of generalizing mathematical structures to the noncommutative setting. Many quantum phenomena have classical counterparts, and can often be modelled by quantized versions of the mathematical structures modelling these classical counterparts. Recently, several mathematical structures have been quantized via a quantization method based on Weaver's notion of a *quantum relation* between von Neumann algebras [10], which he distilled from his work with Kuperberg on the quantization of metric spaces [9]. Quantum relations can be regarded as noncommutative versions of ordinary relations, and admit a rich relational calculus that allows us to generalize concepts to the noncommutative setting. Building on these concepts, Weaver quantized posets [10] and showed that *quantum graphs* [2], which are used for quantum error correction, can be understood in terms of quantum relations [11].

Von Neumann algebras are rather noncommutative generalizations of measure spaces than of sets. Kornell identified *hereditarily atomic* von Neumann algebras, which are essentially (possibly infinite) sums of matrix algebras, as the proper noncommutative generalizations of sets. For this reason, hereditarily atomic von Neumann algebras are also called *quantum sets*, and the category **qRel** of quantum sets and quantum relations can be regarded as the proper noncommutative generalization of the category **Rel** of sets and binary relations. Unlike the category of all von Neumann algebras and quantum relations, **qRel** is dagger compact closed, just like **Rel**. Together with Kornell and Mislove, the second author investigated the categorical properties of quantum posets in this restricted setting of hereditarily atomic von Neumann algebras [8]. Building on this work, they introduced *quantum cpos*, which are noncommutative versions of ω-complete partial orders (cpos). Ordinary cpos can be used to construct denotational models of ordinary programming languages, and in a similar way, they showed that quantum cpos can be used for the denotational semantics of quantum programming languages [7]. Also building on the definition of quantum posets in the hereditarily atomic setting, we introduced *quantum suplattices* [5], which are noncommutative versions of complete lattices and supremum-preserving maps. For the definition of quantum suplattices, the compact structure of **qRel** seems to be essential.

Categorically, quantization via quantum relations can be understood as the internalization of mathematical structures in the category **qRel**, and many theorems about quantized structures via quantum

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relations rely on the categorical properties of **qRel**. There are several categorical generalizations of the category **Rel** such as allegories [3] or bicategories of relations [1], but unfortunately, **qRel** is not an example of either of them. This is mainly due to the fact that the internal functions of **qRel** form a semicartesian monoidal category rather than a cartesian monoidal category, which reflects the quantum character of **qRel**. Tweaking the definitions of either allegories or bicategories of relations is difficult; their cartesian character seems to be essential.

Therefore, we aim to find a different categorical generalization of **Rel** that would capture **qRel**. We take daggers as a primitive notion, and identify six properties of **qRel** as axioms for our categorical generalization of **Rel**. Similar properties also occur in recent categorical axiomatizations of several dagger categories such as the category **Hilb** and **Rel** [4, 6], and likely will form a subset of the axioms of a future categorical characterization of **qRel**. Hence, we define a *semicartesian category of relations* to be a category $\mathbf{R}$ such that

(1) $\mathbf{R}$ is a locally small dagger compact category;
(2) $\mathbf{R}$ has all small dagger biproducts;
(3) $\mathbf{R}$ has precisely two scalars;
(4) $\mathbf{R}$ is a dagger kernel category;
(5) For each object X in $\mathbf{R}$ there is precisely one morphism $X \to I$ with zero kernel;
(6) For each object X and each projection p on X, $p \geq \mathrm{id}_X$ if and only if $\ker p = 0$.

Here, a *projection* on an object X is a morphism $p : X \to X$ such that $p \circ p = p = p^\dagger$. For the last axiom, we use that the first three axioms imply that $\mathbf{R}$ is a *quantaloid*, i.e., a category enriched over the category **Sup** of complete lattices and supremum-preserving maps. As another consequence of the axioms, we prove that the homsets of $\mathbf{R}$ are actually orthomodular lattices. We conclude with a discussion of conditions that assure the existence of a power set construction in semicartesian categories of relations.

References

[1] A. Carboni, R.F.C. Walters, *Cartesian bicategories I*, J. Pure Appl. Alg. 49(1), 11-32, 1987.

[2] R. Duan, S. Severini and A. Winter, *Zero-error communication via quantum channels, noncommutative graphs, and a quantum Lovász number*, IEEE Trans. IT., 59, 1164-1174, 2013.

[3] P. Freyd, A. Scedrov, *Categories, Allegories*, North-Holland, Amsterdam, 1990.

[4] C. Heunen, A. Kornell, *Axioms for the category of Hilbert spaces*, PNAS **119**, e2116285119, 2022.

[5] G. Jenča, B. Lindenhovius, *Quantum Suplattices*, Proceedings of the 20th International Conference on Quantum Physics and Logic, pp. 58–74, 2023.

[6] A. Kornell, *Axioms for the category of sets and relations*, arXiv:2302.14153 [math.CT], 2023.

[7] A. Kornell, B. Lindenhovius, and M. Mislove, *Quantum CPOs*, Proceedings QPL, 2021.

[8] A. Kornell, B. Lindenhovius, and M. Mislove, *A category of quantum posets*, Indagationes Mathematicae **33**(6), 1137–1171, 2022.

[9] G. Kuperberg, N. Weaver, *A von Neumann algebra approach to quantum metrics*, Mem. Amer. Math. Soc. **215**, 2012.

[10] N. Weaver, *Quantum relations*, Mem. Amer. Math. Soc. **215**, 2012.

[11] N. Weaver, *Quantum Graphs as Quantum Relations*, J. Geom. Anal. **31**, 9090–9112, 2021.

Sketches and Classifying Logoi

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Abstract. This talk is based on the preprint [4].

The notion of *sketch* was introduced by Ehrensmann [3, 5]. It consists of a category together with a specification of certain cones and cocones. Using the idea that certain logical operations can be described through limits and colimits, sketches have been considered as one of the many formalisation of the concept of *theory* [2, 7, 8]. In particular, they can be used to present theories in infinitary logics.

The aim of this work is to extend what was done with sites and topoi in the context of geometric logic to infinitary logic introducing the notions of *rounded sketch* and *logos*. More precisely, we want to replicate the pattern that sites are presentations of *geometric* theories and that the classifying topos gives a *syntax independent* avatar of the theory. In a similar way our notion of *rounded sketch* gives the presentation of any infinitary theory (including geometric ones) and the classifying logos its syntax independent presentation.

Logic Fragment	**Presentation**	**Morita Classifying Object**
Geometric	Site	Topos
Infinitary	Rounded Sketch	Logos

We start showing some nice (*topological*) properties of the 2-category of sketches, which turn out to be useful for some important constructions. For instance, we give an explicit formula to calculate weighted pseudo co/limits in the 2-category of sketches and we prove that the tensor product for sketches (studied by Benson in [1]) is closed.

Then, we provide some normalisation constructions which will be useful for our main result, a Diaconescu-like theorem for rounded sketches and logoi. More precisely, for an appropriate notion of *Morita smallness*, we show that for any Morita small sketch $\mathcal{S}$ we can construct its *left sketch classifier* $\hat{\mathcal{S}}$, i.e. a left sketch together with a sketch morphism $J_{\mathcal{S}} \colon \mathcal{S} \to \hat{\mathcal{S}}$ inducing, for any left sketch $\mathcal{M}$, an equivalence as below.

$$- \circ J_{\mathcal{S}} \colon \mathsf{LSkt}(\hat{\mathcal{S}}, \mathcal{M}) \to \mathsf{Skt}(\mathcal{S}, \mathcal{M})$$

Moreover, we use this result to prove that the $\widehat{(-)}$-construction restricted to rounded sketches shows that the 2-category $\mathsf{Log}^{\mathsf{M}}$ of Morita small logoi is (bi)reflective in the 2-category $r\mathsf{Skt}^{\mathsf{M}}$, of Morita small rounded sketches.

$$\begin{tikzcd}
r\mathsf{Skt}^{\mathsf{M}} \arrow[r, bend right, "\mathcal{C}l[-]"'] \arrow[r, phantom, "\top"] & \mathsf{Log}^{\mathsf{M}} \arrow[l, bend right, "U"']
\end{tikzcd}$$

This result generalises similar known ones for classifying topoi and Φ-exact categories [6], summarised in the commutative (not considering the dashed arrows) diagram below.

$$\begin{tikzcd}
\Phi\text{-ex} \arrow[r] \arrow[d, bend left, "P_\Phi"] \arrow[d, phantom, "\vdash"] & \mathsf{MSite} \arrow[r] \arrow[d, bend left, "\mathsf{Sh}"] \arrow[d, phantom, "\vdash"] & r\mathsf{Skt}^{\mathsf{M}} \arrow[d, bend left, "\mathcal{C}l[-]"] \arrow[d, phantom, "\vdash"] \\
\infty\text{-}\mathsf{Ex}^{\mathsf{M}} \arrow[r] \arrow[u, bend left, dashed, "U"] & \mathsf{Topoi}^{\mathrm{op}} \arrow[r] \arrow[u, bend left, dashed, "J"] & \mathsf{Log}^{\mathsf{M}} \arrow[u, bend left, dashed, "J"]
\end{tikzcd}$$

References

[1] D. B. Benson, *Multilinearity of sketches*, Theory and Applications of Categories 3, no. 11, 1997.

[2] J. Adámek and J. Rosický, *Locally Presentable and Accessible Categories*, London Mathematical Society Lecture Note Series, Cambridge University Press, 1994.

[3] A. Bastiani and C. Ehresmann, *Categories of sketched structures*, Cahiers de topologie et géométrie différentielle, 13(2):104–214, 1972.

[4] I. Di Liberti and G. Lobbia, Sketches and Classifying Logoi, preprint available at arXiv:2403.09264, 2024.

[5] C. Ehresmann, *Esquisses et types des structures algébriques*, Bul. Inst. Politehn. Iaşi (N.S.), 14/18:1–14, 1968.

[6] R. Garner and S. Lack, *Lex colimits*, Journal of Pure and Applied Algebra 216, no. 6, 2014.

[7] M. Makkai and R. Paré, *Accessible Categories: The Foundations of Categorical Model Theory*, American Mathematical Society, 1989.

[8] C. Wells, *Sketches: Outline with references*, Dept. of Computer Science, Katholieke Universiteit Leuven Celestijnenlaan A, 200, 1994.

Bicategories for automata theory

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Abstract.
It has long been known [EKKK74] that automata can be interpreted within every monoidal category $(\mathcal{K}, \otimes, I)$; the cornerstone results in this direction are essentially three:

S1. if $T : \mathcal{K} \to \mathcal{K}$ is a commutative monad, 'Mealy' and 'Moore' machines in the (monoidal) Kleisli category $\mathcal{K}_T$ are 'non-deterministic' machines for a notion of fuzziness prescribed by T (examples of this are: exceptions monads, various probability monads, the powerset monad);

S2. if $\mathcal{K}$ is closed, one can characterize Mealy and Moore machines coalgebraically [Jac06], and this provides a slick proof of the cocompleteness of the categories $\mathbf{Mly}(A, B)$ and $\mathbf{Mre}(A, B)$ that they form [AT90];

S3. if (and curiously enough, *only if*) $\mathcal{K}$ is Cartesian monoidal, $\mathbf{Mly}(A, B)$ is the hom-category of a bicategory **Mly** [Gui74, KSW97], and $\mathbf{Mre}(A, B)$ the hom-category of a *semibicategory* (a bicategory without identity 1-cells, cf. [Mit72, MBCB02] and [BFL+23]) **Mre**.

Starting from the well-known principle that regards a monoidal category as nothing but a single-object bicategory, we fix a general bicategory $\mathcal{B}$ and study 'abstract machines' in $\mathcal{B}$, i.e. diagrams of 2-cells of the form

$$\begin{tikzcd}
& X \arrow[d, "i"'] \arrow[dd, bend right=80, "o"'] \arrow[dd, bend left=80, "e"] & \\
{} \arrow[r, phantom, "\underset{\sigma}{\Leftarrow}"] & X \arrow[d, "e"'] \arrow[r, phantom, "\overset{\delta}{\Rightarrow}"] & {} \\
& Y &
\end{tikzcd}$$

where i, e, o are 1-cells respectively dubbed the 'input' 1-cell, the 'state' 1-cell and the 'output' 1-cell.
We then proceed to find parallels for S1, S2, S3 in this more general setting:

B1. let T be a monad on **Set** and $(V, \odot, \perp)$ a quantale. The study of bicategorical machines in the bicategory of (T, V)-relations of [HST14] accounts for notions of non-determinism that are modeled on topologies, approach structures, metric and ultrametric structures, Kuratowski closure spaces, and all the likes of structures studied by monoidal topology;

B2. in perfect parallel with the monoidal case, the *behaviour* of a Mealy/Moore machine can be characterized through a universal property [Gog72]; a terminal coalgebra for monoidal machines, a weighted limit of sorts for bicategorical machines. In the case of Moore machines the description is prettier, in terms of a (pointwise) right extension. This clarifies long-forgotten remarks by Bainbridge [Bai75] on properties of abstract machines seen as Kan extensions;

B3. passing from single- to multi-object bicategories, we gain an additional degree of freedom by indexing hom-categories over generic objects; in particular, we gain a rich compositional structure that was not present in the monoidal case, a way of composing machines that is neither sequential nor parallel and that we dub *intertwining*.

This talk presents, and expands on, joint works with A. Laretto, G. Boccali, B. Femić, S. Luneia, see [BLLL23, BFL+23]

References

[AT90] J. Adámek and V. Trnková, *Automata and algebras in categories*, Kluwer, 1990.

[Bai75] E.S. Bainbridge, *Addressed machines and duality*, Category Theory Applied to Computation and Control (Berlin, Heidelberg) (Ernest Gene Manes, ed.), Springer Berlin Heidelberg, 1975, pp. 93–98.

[BFL+23] G. Boccali, B. Femić, A. Laretto, F. Loregian, and S. Luneia, *The semibicategory of moore automata*, 2023.

[BLLL23] Guido Boccali, Andrea Laretto, Fosco Loregian, and Stefano Luneia, *Bicategories of automata, automata in bicategories*, Electronic Proceedings in Theoretical Computer Science **397** (2023), 1–19.

[EKKK74] H. Ehrig, K.-D. Kiermeier, H.-J. Kreowski, and W. Kühnel, *Universal theory of automata. A categorical approach*, Teubner Studienbücher. Informatik. Stuttgart, 1974 (English).

[Gog72] J. Goguen, *Realization is universal*, Mathematical systems theory **6** (1972), 359–374.

[Gui74] R. Guitart, *Remarques sur les machines et les structures*, Cahiers de Topologie et Géométrie Différentielle Catégoriques **15** (1974), 113–144.

[HST14] D. Hofmann, G.J. Seal, and W. Tholen, *Monoidal topology: A categorical approach to order, metric, and topology*, vol. 153, Cambridge University Press, 2014.

[Jac06] B. Jacobs, *A bialgebraic review of deterministic automata, regular expressions and languages*, Algebra, Meaning, and Computation, Springer Berlin Heidelberg, 2006, pp. 375–404.

[KSW97] P. Katis, N. Sabadini, and R.F.C. Walters, *Bicategories of processes*, Journal of Pure and Applied Algebra **115** (1997), no. 2, 141–178.

[MBCB02] M.-A. Moens, U. Berni-Canani, and F. Borceux, *On regular presheaves and regular semicategories*, Cahiers de Topologie et Géométrie Différentielle Catégoriques **43** (2002), no. 3, 163–190 (en). MR 1928230

[Mit72] B. Mitchell, *The dominion of isbell*, Transactions of the American Mathematical Society **167** (1972), no. 0, 319–331.

Doubly-infinitary distributive categories

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Abstract.

A common question in category theory is how limits and colimits interact with each other. One of the most benign kinds of interaction is that of a *(pseudo)distributive law*; for instance, finitary and infinitary *distributive categories* [1], and *completely distributive categories* [4].

In [3], we explore the realm of categories with products and coproducts, featuring a distributive law between them, which we term doubly-infinitary distributive categories. This notion serves as an intermediary between infinitary distributive categories and completely distributive ones.

We show various instances of doubly-infinitary distributive categories aiming for a comparative analysis with established notions such as extensivity, infinitary distributiveness, and cartesian closedness. Our exploration reveals that this condition represents a substantial extension beyond the classical understanding of infinitary distributive categories. We also remark that free doubly-infinitary distributive categories are cartesian closed.

In this talk, we intend to address some of these insights. The talk is mostly based on [2, 3] and ongoing work.

References

[1] A. Carboni, S. Lack, and R. F. Walters. Introduction to extensive and distributive categories. Journal of Pure and Applied Algebra, 84(2):145–158, 1993.

[2] Fernando Lucatelli Nunes and Matthijs Vákár. Σ-tractible coproducts and Cartesian closure of Grothendieck constructions. To appear.

[3] Fernando Lucatelli Nunes and Matthijs Vákár. Free Doubly-Infinitary Distributive Categories are Cartesian Closed. arXiv:2403.10447, 2024

[4] F. Marmolejo, R. Rosebrugh, and R. Wood.Completely and totally distributive categories i. Journal of Pure and Applied Algebra, 216(8-9):1775–1790, 2012.

The Giry monad revisited

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Abstract.
The thesis of this contribution is that one does not have to restrict the class of measurable spaces under consideration to develop a fruitful theory of probability. To this end we suggest a variation of the concept of the Giry monad. As in Giry's legendary paper, we motivate the new concept by study of preservation of limits contributing new results. As an application we show that the Wasserstein distance is actually a distance on the probability measures considered in our set-up.

Motivation.
In 1982 Giry introduced her concept of Giry monad in two variations—first, as a monad on the category <u>Meas</u> of measurable spaces[1] and, second, as a monad on the category of Polish spaces. Both are motivated with preservation of limit properties, which turn out to be stronger in the later case.

Around the same time, some deep exploration of measure theory was still taking place: [1, 2, 3, 5]. Unfortunately, it seems that the relevance of the remarkable and miraculous result of Pachl [2] for a categorical approach to probability was not spotted. Actually, it enables one, to generalise Giry's results for Polish spaces to measurable spaces.

When working with general measurable spaces and measures thereon some limitations occur:

1. projections of measurable sets are not necessarily measurable
2. the Giry monad does not necessarily (weakly) preserve directed limits[2], and
3. countably generated σ-algebras are normally too small to model the notion of "almost surely".

Classically, these issues are addressed by restricting the class of spaces under consideration—with analytic spaces (including Polish spaces) being the most general class that allows for a rich theory (actually, this approach basically transfers limitation 1 into a definition). But analytic spaces do not encompass the theory of distributions, i.e. discrete measures on arbitrary sets, which play a paramount role in logic and computer science. A solution to limitation 3 is to enlarge the σ-algebra by completing it. Though this process comes at a price: Since a countable representation is lost, one is often forced into a situational choice.

Another problem is that quite natural "large" examples are excluded by the classical approach, e.g. the measurable space induced by the well-known French railway metric defined on the set $\mathbb{R}^2$:

$$d(u,v) = \begin{cases} ||u| - |v|| & \text{if } u = rv \text{ for some } r \in (0,\infty) \\ |u| + |v| & \text{else} \end{cases} \qquad \text{for } u, v \in \mathbb{R}^2 \tag{1}$$

[1] i.e. pairs $(X, \mathcal{A})$ of a set X and a σ-algebra $\mathcal{A}$

[2] in elementary terms: given consistent probabilities on the objects of a diagram in <u>Meas</u> a probability on the limit need not exist

modelling all potential railway lines in France. As for more categorical limitations, note that analytic spaces can have at most the cardinality of the continuum, so arbitrary limits and colimits are already excluded by size.

We suggest to remedy the situation as follows: Instead of excluding certain measurable spaces, we restrict the Giry monad. Namely, let a **law** on a measurable space $(X, \mathcal{A})$ be a probability measure thereon, such that it extends to a probability measure p' on a larger set $X_p \supseteq X$ on which a σ-algebra $\mathcal{A}_p$ generated from a semicompact paving[3] is given such that $\mathcal{A}$ is a subset of the p'-completion of $\mathcal{A}_p$. This approach generalises the notion of a Radon space. Probably, this idea appeared to one or the other already. But they then failed to form a functor therefrom, as one must guarantee that the push-forward of a law along a measurable map is again a law. Surprisingly, Pachl proved this in 1979 [2] (see also [6, 452R]). So we define the **Giry monad** on $(X, \mathcal{A})$ to be the collection of laws on $(X, \mathcal{A})$.

Results.

As directed limits are not preserved by the Giry monad as defined by Giry [4], she had to impose a technical condition. Only in the case of Polish spaces, which she discusses only for the index set ω^{op}, she could avoid this technical condition. We prove limit preservation in the case of our Giry monad holds for general directed index categories on measurable spaces.

Moreover, we discuss other limit shapes, especially pushouts, where we can provide a result of weak limit preservation. In this context, we can also show that the Wasserstein distance is a distance, i.e. satisfies the triangle inequality, for laws. Classically, the Wasserstein distance is only considered for probability measures on separable metric spaces, where it is a corner stone of several industries—e.g. optimal transport or concurrency theory in computer science. To round up the discussion, we give a negative examples of shapes that are not preserved, e.g. equalisers.

For analytic spaces, our Giry monad coincides with Giry's original definition. The same holds in the example expressed in (1). We also add a few propositions paving the way to a further development of probability (and measure) theory based on the current suggestion.

We conclude with some set theoretic remarks. Moreover, we give some thoughts on how to "extend" our approach to analytic spaces, i.e. define a measure that looks like a measure on an analytic space.

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References

[1] J.K. Pachl, *Disintegration and compact measures*, Math. Scand. (1978), no. 43, pp. 157–168.

[2] J.K. Pachl, *Two classes of measures*, Colloq. Math. (1979), vol. 1, pp. 331–340.

[3] N. Falkner, *Generalizations of analytic and standard measurable spaces*, Math. Scand. (1981), pp. 283–301.

[4] M. Giry, *A categorical approach to probability*, in *Categorical Aspects of Topology and Analysis*, Lecture Notes in Math., vol. 915, Springer, 1982

[5] R. Shortt, *Universally measurable spaces: an invariance theorem and diverse characterizations*, Fund. Math. (1984), no. 121(2), pp. 169–176.

[6] H. Fremlin, *Measure Theory*, vol. 4, Torres Fremlin, 2003.

[3] also called *countably compact* or *semicompact class*, a collection of subsets of X such that every countable subcollection satisfying the finite intersection property (*FIP*, i.e. every finite subcollection has non-empty intersection) has non-empty intersection

$\mathscr{V}$-graded categories and $\mathscr{V}$-$\mathscr{W}$-bigraded categories: Functor categories and bifunctors over non-symmetric bases

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Abstract. Categories graded by a monoidal category $\mathscr{V}$, or ($\mathscr{V}$-)*graded categories*, were introduced by Wood [1] under the name *large $\mathscr{V}$-categories*, and they simultaneously generalize both $\mathscr{V}$-enriched categories and $\mathscr{V}$-actegories in the absence of any assumptions on $\mathscr{V}$; also see [2, 3, 4]. Explicitly, $\mathscr{V}$-graded categories may be defined as categories enriched in the presheaf category $\hat{\mathscr{V}} = [\mathscr{V}^{\mathsf{op}}, \mathrm{SET}]$ with its Day convolution monoidal structure, while they also admit a direct elementwise definition.

Given an arbitrary strict monoidal category $\mathscr{V}$, we show that $\mathscr{V}$-graded categories support a robust theory of graded functor categories and bifunctors, enabled by a notion of *bigraded category* that we introduce. This is in contrast with the usual settings of enriched category theory, where the definition of enriched functor categories and bifunctors employs a symmetry [5], braiding, or duoidal structure on $\mathscr{V}$ [6] and so is not applicable when working with just a biclosed monoidal category $\mathscr{V}$ or, more generally, a closed bicategory, where one nevertheless has a robust theory of $\mathscr{V}$-modules and of $\mathscr{V}$-categories of $\mathscr{V}$-valued presheaves [7, 8], but these are not defined in terms of bifunctors and functor categories.

We develop our results on graded functor categories in a general setting that begins with a given pair of strict monoidal categories $\mathscr{V}$ and $\mathscr{W}$. Writing $\mathscr{W}^{\mathsf{rev}}$ to denote the reverse of $\mathscr{W}$, we consider both $\mathscr{V}$-graded categories and $\mathscr{W}^{\mathsf{rev}}$-graded categories, calling the former *left $\mathscr{V}$-graded categories* and the latter *right $\mathscr{W}$-graded categories*. A *$\mathscr{V}$-$\mathscr{W}$-bigraded category* is then a left $(\mathscr{V} \times \mathscr{W}^{\mathsf{rev}})$-graded category and so has both an underlying left $\mathscr{V}$-graded category and an underlying right $\mathscr{W}$-graded category. For example, both $\mathscr{V}$ and $\hat{\mathscr{V}}$ underlie $\mathscr{V}$-$\mathscr{V}$-bigraded categories.

Given a left $\mathscr{V}$-graded category $\mathscr{A}$ and a $\mathscr{V}$-$\mathscr{W}$-bigraded category $\mathscr{C}$, we show that there is a right $\mathscr{W}$-graded category $[\mathscr{A}, \mathscr{C}] = {}^{\mathscr{V}}[\mathscr{A}, \mathscr{C}]_{\mathscr{W}}$ whose objects are (left) $\mathscr{V}$-graded functors from $\mathscr{A}$ to $\mathscr{C}$. Similarly, given a right $\mathscr{W}$-graded category $\mathscr{B}$ and a $\mathscr{V}$-$\mathscr{W}$-bigraded category $\mathscr{C}$, we obtain a left $\mathscr{V}$-graded category $[\mathscr{B}, \mathscr{C}] = {}_{\mathscr{V}}[\mathscr{B}, \mathscr{C}]^{\mathscr{W}}$ whose objects are right $\mathscr{W}$-graded functors from $\mathscr{B}$ to $\mathscr{C}$. In particular, if $\mathscr{D}$ is a left $\mathscr{V}$-graded category, then its *opposite* $\mathscr{D}^{\mathsf{op}}$ is a *right* $\mathscr{V}$-graded category, so if $\mathscr{C}$ is a $\mathscr{V}$-$\mathscr{V}$-bigraded category then $[\mathscr{D}^{\mathsf{op}}, \mathscr{C}]$ is a left $\mathscr{V}$-graded category.

Given a left $\mathscr{V}$-graded category $\mathscr{A}$ and a right $\mathscr{W}$-graded category $\mathscr{B}$, we construct a $\mathscr{V}$-$\mathscr{W}$-bigraded category $\mathscr{A} \boxtimes \mathscr{B}$ whose objects are pairs (A, B) with $A \in \mathsf{ob}\,\mathscr{A}$ and $B \in \mathsf{ob}\,\mathscr{B}$. Given also a $\mathscr{V}$-$\mathscr{W}$-bigraded category $\mathscr{C}$, we may therefore consider $\mathscr{V}$-$\mathscr{W}$-bigraded functors of the form $F : \mathscr{A} \boxtimes \mathscr{B} \to \mathscr{C}$, which provide a notion of *bifunctor* in the graded setting. Writing ${}_{\mathscr{V}}\mathrm{GCAT}$, $\mathrm{GCAT}_{\mathscr{W}}$, and ${}_{\mathscr{V}}\mathrm{GCAT}_{\mathscr{W}}$ for the 2-categories of left $\mathscr{V}$-graded categories, right $\mathscr{W}$-graded categories, and $\mathscr{V}$-$\mathscr{W}$-bigraded categories, respectively, we show that there are 2-natural isomorphisms

$$ {}_{\mathscr{V}}\mathrm{GCAT}(\mathscr{A}, [\mathscr{B}, \mathscr{C}]) \cong {}_{\mathscr{V}}\mathrm{GCAT}_{\mathscr{W}}(\mathscr{A} \boxtimes \mathscr{B}, \mathscr{C}) \cong \mathrm{GCAT}_{\mathscr{W}}(\mathscr{B}, [\mathscr{A}, \mathscr{C}]) \; . $$

In the special case where $\mathscr{V}$ is *symmetric* monoidal and we take $\mathscr{W} = \mathscr{V}$, there is no essential distinction between left and right $\mathscr{V}$-graded categories, while every $\mathscr{V}$-graded category is canonically $\mathscr{V}$-$\mathscr{V}$-bigraded, and we recover the $\mathscr{V}$-graded functor categories and bifunctors that were studied by Wood [1, §1.6] and coincide with the usual $\hat{\mathscr{V}}$-enriched concepts for the symmetric monoidal category $\hat{\mathscr{V}} = [\mathscr{V}^{\mathsf{op}}, \mathrm{SET}]$, though $\mathscr{A} \boxtimes \mathscr{B}$ does not coincide with the monoidal product of $\hat{\mathscr{V}}$-categories $\mathscr{A} \otimes \mathscr{B}$.

Given an arbitrary strict monoidal category $\mathscr{V}$ and a pair of *right* $\mathscr{V}$-graded categories $\mathscr{A}$ and $\mathscr{B}$, we may consider $\mathscr{V}$-$\mathscr{V}$-bigraded functors $F : \mathscr{B}^{\mathsf{op}} \boxtimes \mathscr{A} \to \mathscr{C}$ valued in any $\mathscr{V}$-$\mathscr{V}$-bigraded category $\mathscr{C}$, and we call these *$\mathscr{V}$-graded modules* from $\mathscr{A}$ to $\mathscr{B}$ in $\mathscr{C}$. Passing to the special case where $\mathscr{C} = \hat{\mathscr{V}}$, we show that $\mathscr{V}$-graded modules in $\hat{\mathscr{V}}$ are precisely $\hat{\mathscr{V}}$-modules between $\hat{\mathscr{V}}$-categories, in the sense obtained by specializing [7, 8] to base of enrichment $\hat{\mathscr{V}} = [\mathscr{V}^{\mathsf{op}}, \mathrm{SET}]$. Furthermore, we show that the $\hat{\mathscr{V}}$-enriched presheaf $\hat{\mathscr{V}}$-category $\mathcal{P}\mathscr{B}$ that is obtained by applying Street's enriched presheaf construction [7] relative to the base of enrichment $\hat{\mathscr{V}}$ is precisely the right $\mathscr{V}$-graded category $[\mathscr{B}^{\mathsf{op}}, \hat{\mathscr{V}}]$ that is obtained as an example of the above general construction of graded functor categories.

References

[1] R. J. Wood, *Indicial methods for relative categories*, Ph.D. thesis, Dalhousie University, 1976.

[2] R. Garner, *An embedding theorem for tangent categories*, Adv. Math. **323** (2018), 668–687.

[3] P. B. Levy, *Locally graded categories*, Talk slides, https://www.cs.bham.ac.uk/∼pbl/papers/locgrade.pdf, 2019.

[4] D. McDermott and T. Uustalu, *Flexibly graded monads and graded algebras*, Lecture Notes in Comput. Sci., vol. 13544, Springer, 2022, pp. 102–128.

[5] G. M. Kelly, *Basic concepts of enriched category theory*, Repr. Theory Appl. Categ. (2005), no. 10, Reprint of the 1982 original [Cambridge Univ. Press].

[6] R. Garner and I. López Franco, *Commutativity*, J. Pure Appl. Algebra (2016), 1707–1751.

[7] R. Street, *Enriched categories and cohomology*, Repr. Theory Appl. Categ. (2005), no. 14, 1–18, Reprinted from Quaestiones Math. **6** (1983), no. 1-3, 265–283.

[8] R. Betti, A. Carboni, R. Street, and R. Walters, *Variation through enrichment*, J. Pure Appl. Algebra **29** (1983), no. 2, 109–127.

Slack Hopf monads

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Abstract.

In this talk, based on [1], we introduce *slack Hopf monads* and investigate their connection to the quasi-Hopf algebras of Drinfel′d. Opmonoidal monads are monads compatible with the monoidal structure: their Eilenberg-Moore category is monoidal and the forgetful functor is strong monoidal [2]. A left Hopf monad is an opmonoidal monad satisfying an extra rigidity condition that on a monoidal left closed category ensures that: (a) the Eilenberg-Moore category is left closed too, and; (b) the forgetful functor is a strong closed functor [3]. In slack left Hopf monads the condition (b) is relaxed to an (essentially unique) slack Hopf structure. Furthermore, the condition of monoidality is dropped, requiring only a magma category. This allows to capture the monads induced by tensoring with a quasi-Hopf algebra (these are not necessarily opmonoidal nor Hopf). We characterise quasi-Hopf algebras among those quasi-bialgebras that induce slack Hopf monads.

References

[1] A. Bruguières, M. Haim, I. López Franco, *Slack Hopf monads*, J. of Algebra 642 (2024) 1–48.

[2] I. Moerdijk, *Monads on tensor categories*, J. Pure Appl. Algebra 168 (2–3) (2002) 189–208.

[3] A. Bruguières, S. Lack, A. Virelizier, *Hopf monads on monoidal categories*, Adv. Math. 227 (2) (2011) 745–800.

Weak equivalences between algebraic weak ω-categories

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Abstract.
A *weak ω-category* in Leinster's sense [1] (see also Batanin [2]) has "all" the operations that a strict ω-category has, but need not satisfy any of the relations. In this setup, one can generalise certain arguments concerning strict ω-categories to weak ones by *encoding relations as operations.*

For example, consider the *unit law.* Given a 1-cell $f\colon x \to y$ in a weak ω-category, we can make sense of the expression $f \circ 1_x$ (because we have all the operations, including identities and compositions), but the relation $f \circ 1_x = f$ does not necessarily hold. Instead, we can consider the operation in a strict ω-category that takes a 1-cell f and spits out a(n identity) 2-cell $f \circ 1_x \to f$. This operation can be lifted to the weak ω-category, and (using a result [3] presented at CT2023) one can check that the resulting 2-cell is invertible in a suitable sense, establishing a kind of unit law.

In this talk, I will describe how to make use of such encoding and prove that the class of *weak equivalences* (an ω-dimensional version of essentially surjective, fully faithful functors) enjoys the *2-out-of-3 property*, i.e. if any two of F, G and GF are weak equivalences then so is the third, generalising the strict case treated in [4].

References

[1] T. Leinster. *Higher operads, higher categories.* Volume 298 of London Mathematical Society Lecture Note Series. Cambridge University Press, 2004. 380pp.

[2] M.A. Batanin. *Monoidal globular categories as a natural environment for the theory of weak n-categories.* Advances in Mathematics 136(1), 39–103, 1998.

[3] S. Fujii, K. Hoshino, and Y. Maehara. *Weakly invertible cells in a weak ω-category.* To appear in Higher Structures. arXiv:2303.14907.

[4] Y. Lafont, F. Métayer, and K. Worytkiewicz. *A folk model structure on omega-cat.* Advances in Mathematics 224(3):1183–1231, 2010.

Internalization of decorated bicategories via π_2-indexings

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Abstract.

Given a bicategory $\mathcal{B}$, and a category $\mathcal{B}^*$, such that the collections of objects of $\mathcal{B}$ and $\mathcal{B}^*$ are equal, we wish to construct interesting double categories $\mathbb{D}$ having $\mathcal{B}$ as horizontal bicategory, and having $\mathcal{B}^*$ as category of objects. We say that the pair $(\mathcal{B}^*, \mathcal{B})$ is a decorated bicategory and $\mathbb{D}$ is an internalization of $(\mathcal{B}^*, \mathcal{B})$. The problem of understanding internalizations of decorated bicategories has been considered in the series of papers [1, 2, 3], where the definition of a numerical invariant, called the vertical length $\ell\mathbb{D}$, associated to every double category $\mathbb{D}$, was introduced. Roughly, the number $\ell\mathbb{D}$ measures the amount of work one would be expected to do to construct a generic square in $\mathbb{D}$, from squares in $(\mathbb{D}_0, H\mathbb{D})$. 1 is the minimum possible length of a double category, and most double categories in the literature, e.g. $\mathbb{M}\mathbf{od}, \mathbb{P}\mathbf{rof}, \mathbb{B}\mathbf{ord}, \mathbb{A}\mathbf{dj}$ are of length 1.

The particual problem of deciding whether a decorated bicategory $(\mathcal{B}^*, \mathcal{B})$ admits internalizations of length 1 has been study in [4]. We present a type of structure allowing to construct internalizations of lenght 1. We call the structure we study π_2-indexings, which are a type of indexing associated to a decorated bicategory $(\mathcal{B}^*, \mathcal{B})$, relating the arrows of $\mathcal{B}^*$ with 2-cells of a specific type in $\mathcal{B}$. The goal of the talk is to present the main results in [4], examples and conjectures related to the problem.

References

[1] J. Orendain. *Lifting bicategories into double categories: The globularily generated condition.* Theory and Applications of Categories, 34:80–108, 2019.

[2] J. Orendain. *Free globularly generated double categories.* Cahiers de topologie et géometrie différentielle catégoriques, LXII Issue 3:243–302, 2021.

[3] J. Orendain. *Free globularly generated double categories ii: The canonical double projection.* Theory and Applications of Categories, 34(42):1343–1385, 2019.

[4] J. Orendain, J.R. Maldonado-Herrera, Internalization of decorated bicategories via π_2-indexings, preprint arXiv:2310.18673, 2023.

On the representability of actions of non-associative algebras

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Abstract.
It is well known that in the semi-abelian category **Grp** of groups, internal actions are represented by automorphisms. This means that the category **Grp** is *action representable* and the representing object, which is called the *actor*, is the group of automorphisms. Another example of action representable category is the variety **Lie** of Lie algebras over a fixed field $\mathbb{F}$, with the actor of a Lie algebra $\mathfrak{g}$ being the Lie algebra of derivations $\mathrm{Der}(\mathfrak{g})$. The notion of action representable category has proven to be quite restrictive: for instance, if a non-abelian variety $\mathcal{V}$ of non-associative algebras over an infinite field $\mathbb{F}$, with $\mathrm{char}(\mathbb{F}) \neq 2$, is action representable, then $\mathcal{V} = \mathbf{Lie}$. More recently G. Janelidze introduced the notion of *weakly action representable category*, which includes a wider class of categories, such as the variety **Assoc** of associative algebras and the variety **Leib** of Leibniz algebras.

In this talk we show that for an *algebraically coherent* and *operadic* variety $\mathcal{V}$ and an object X of $\mathcal{V}$, it is always possible to construct a *partial algebra* $\mathcal{E}(X)$, called *external weak actor* of X, and a natural monomorphism of functors

$$\tau\colon \mathrm{Act}(-,X) \rightarrowtail \mathrm{Hom}_{\mathbf{PAlg}}(U(-),\mathcal{E}(X)),$$

where **PAlg** is the category of partial algebras over $\mathbb{F}$ and $U\colon \mathcal{V} \to \mathbf{PAlg}$ denotes the forgetful functor. The pair $(\mathcal{E}(X),\tau)$ is called *external weak representation* of the functor $\mathrm{Act}(-,X)$. Moreover, for any other object B of $\mathcal{V}$, we provide a complete description of the morphisms $(B \to \mathcal{E}(X)) \in \mathrm{Im}(\tau_B)$, i.e. of the homomorphisms of partial algebras which identify the actions of B on X in $\mathcal{V}$, and we show that the existence of a weak representation is closely connected to the *amalgamation property*, which we use to prove that the variety **CAssoc** of commutative associative algebras is weakly action representable.

Eventually, we give an application of the construction of the external weak actor in the context of varieties of *unital* algebras, which are *ideally exact categories* in the sense of G. Janelidze: we prove that, if $\mathcal{V} = \mathbf{Alt}$ is the variety of *alternative* algebras and X is a unital alternative algebra, then $\mathcal{E}(X) \cong X$ is the actor of X. In other words, unital alternative algebras, such as the algebra $\mathbb{O}$ of *octonions*, have representable actions.

This is joint work with J. Brox, (*Universidad de Valladolid*, Spain), Alan S. Cigoli (*Università degli Studi di Torino*, Italy), Xabier García Martínez (*Universidade de Vigo*, Spain), Giuseppe Metere (*Università degli Studi di Milano*, Italy), Tim Van der Linden and Corentin Vienne (*Université catholique de Louvain*, Belgium).

References

[1] A. S. Cigoli, M. Mancini M. and G. Metere, "On the representability of actions of Leibniz algebras and Poisson algebras", *Proceedings of the Edinburgh Mathematical Society* **66** (2023), no. 4, pp. 998–1021.

[2] J. Brox, X. García-Martínez, M. Mancini, T. Van der Linden and C. Vienne, "Weak representability of actions of non-associative algebras" (2024), submitted, preprint available at `arXiv:2306.02812`.

[3] G. Janelidze, "Central extensions of associative algebras and weakly action representable categories", *Theory and Applications of Categories* **38** (2022), No. 36, pp 1395-1408.

[4] G. Janelidze, "Ideally exact categories" (2023), preprint available at `arXiv:2308.06574`.

Opposites and hom weak ω-categories

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Abstract. This talk is based on our recent preprint [1]. We work with globular weak ω-categories, using a recent formulation for them and their computads proposed by Dean et al. [3]. We define opposites of a weak ω-category, changing the direction of all cells whose dimension belongs to a given set. We also give an alternative construction of the hom ω-categories of an ω-category to that of Cottrell and Fujii [2]. We then show that it has an left adjoint and that it preserves the property of being cofibrant.

Computads are structures out of which one can generate weak ω-categories. They consist of sets of generators together with attachment maps, assigning a source and target to each generator. In Dean et al. [3], first, the category Comp of computads and morphisms of ω-categories is defined inductively on dimension, together with an adjunction with the category Glob of globular sets

$$\mathsf{Glob} \underset{\mathsf{Cell}}{\overset{\mathsf{Free}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{Comp}\ .$$

Here, the functor Cell takes a computad to the underlying globular set of the ω-category that it generates, the elements of which are either generators, or formal applicatons of operations of weak ω-categories (compositions and coherences). Then ω-categories are defined as the algebras for the induced monad on globular sets, which was shown to agree with that of Batanin and Leinster [5]. It was further shown by Garner [4] that ω-categories generated by a computad are the cofibrant objects for certain weak factorisation system.

We use a similar technique to construct the opposites and the homs of an ω-category. We start with an adjunction on the level of globular sets:

$$\mathsf{Glob} \underset{\mathsf{op}}{\overset{\mathsf{op}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{Glob} \qquad\qquad \mathsf{Glob} \underset{\Omega}{\overset{\Sigma}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{Glob}_{\star\star}$$

where $\mathsf{Glob}_{\star\star}$ is the category of globular sets with two chosen objects. The functor op is defined with respect to a set of dimensions $w \subset \mathbb{N}_{>0}$, by swapping the source and target of every element whose dimension belongs to w. The functor Ω takes a globular set with two chosen objects to the globular sets of elements between those objects, and the suspension ΣX is the globular set with two objects, such that $\Omega(\Sigma X) = X$. We then observe that in both cases the left adjoint preserves the globular pasting diagrams, which are the arities of the operations of weak ω-categories. Using this observation, we define the opposite and the suspension of a computad together with natural transformations, as shown in the diagrams below:

$$
\begin{tikzcd}
\mathsf{Glob} \arrow[r, "\mathsf{Free}"] \arrow[d, "\mathsf{op}"'] \arrow[dr, phantom, "="] & \mathsf{Comp} \arrow[r, "\mathsf{Cell}"] \arrow[d, "\mathsf{op}"] & \mathsf{Glob} \arrow[d, "\mathsf{op}"] \arrow[dl, Rightarrow, shorten=4mm, "\sim"'] \\
\mathsf{Glob} \arrow[r, "\mathsf{Free}"'] & \mathsf{Comp} \arrow[r, "\mathsf{Cell}"'] & \mathsf{Glob}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Glob} \arrow[r, "\mathsf{Free}"] \arrow[d, "\Sigma"'] \arrow[dr, phantom, "="] & \mathsf{Comp} \arrow[r, "\mathsf{Cell}"] \arrow[d, "\Sigma"] & \mathsf{Glob} \arrow[d, "\Sigma"] \arrow[dl, Rightarrow, shorten=4mm] \\
\mathsf{Glob}_{\star\star} \arrow[r, "\mathsf{Free}"'] & \mathsf{Comp}_{\star\star} \arrow[r, "\mathsf{Cell}"'] & \mathsf{Glob}_{\star\star}
\end{tikzcd}
$$

The mates of those natural transformations under the respective adjunctions give rise to morphisms of monads, hence they lift as functors $\mathsf{op} : \omega\,\mathsf{Cat} \to \omega\,\mathsf{Cat}$ and $\Omega : \omega\,\mathsf{Cat}_{\star\star} \to \omega\,\mathsf{Cat}$. The functor op is an equivalence of categories with itself as its inverse, and by the adjoint lifting theorem, the functor Ω admits a left adjoint Σ.

Finally, we show that the functors op, Ω and Σ preserve the cofibrant objects, by describing a recognition principle for free ω-categories on a computad. We also show that the opposite of a hom ω-category is the hom of the some opposite of the original ω-category, as expected.

References

[1] T. Benjamin and I. Markakis, Opposites of weak ω-categories and the suspension and hom adjunction. preprint arXiv:2402.01611, 2024.

[2] T. Cottrell and S. Fujii, *Hom weak ω-categories of a weak ω-category.* Mathematical Structures in Computer Science, 2022, 32(4):420-441. doi:10.1017/S0960129522000111.

[3] C. Dean, E. Finster, I. Markakis, D. Reutter and J. Vicary, Computads for weak ω-categories as an inductive type, preprint arXiv:2208.08719, 2022.

[4] R. Garner, *Homomorphisms of higher categories*, Advances in Mathematics, 2010, 224(6):2269-2311. doi:10.1016/j.aim.2010.01.022.

[5] T. Leinster, *Higher Operads, Higher Categories*, Cambridge University Press, 2004.

Limit-sketchable infinity categories

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Abstract.
A presentable ∞-category is an accessible localization of an ∞-category of presheaves over some small ∞-category. Presentable ∞-categories play a key role in the study of higher topoi [2], stable ∞-categories and higher algebra. In ordinary category theory, a limit sketch is a categorical formalization of the notion of an essentially algebraic theory. The Representation Theorem of Adámek and Rosický [1] states that locally presentable categories are equivalent to categories of models of limit sketches.

In our research, we prove an analogous representation theorem in the context of ∞-categories, by showing that an ∞-category is presentable if and only if it is limit-sketchable. Moreover, we show that numerous ∞-categories, including complete Segal spaces, ∞-operads, E_∞-algebras, spectra, and infinite loop spaces, can be constructed as ∞-categories of models of limit sketches. Our representation theorem yields explicit presentable structures underlying each of the examples that we consider. This is joint work with Carles Casacuberta and Javier J. Gutiérrez.

References

[1] J. Adámek and J. Rosický. *Locally Presentable and Accessible Categories.* Vol. 189. Cambridge University Press, 1994

[2] J. Lurie. *Higher Topos Theory.* Annals of Mathematics Studies 170. Princeton University Press, 2009.

No-iteration transitions and no-iteration distributive laws for pseudomonads

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Abstract.
The distributive laws for monads were introduced by J. Beck in [Beck, 1969], then years later these ideas were extended to high dimensional categories. F. Marmolejo defined a distributive law between pseudomonads in [Marmolejo, 1999] and the four 2–cells requested in this definition (two triangles and two pentagons) are subject to nine coherence conditions.
A few years later in the thesis [Tanaka, 2005], an extra coherence condition apart from the nine described in the paper of F. Marmolejo was given, but in [Marmolejo and Wood, 2008], F. Marmolejo and R. Wood showed that the extra condition in the thesis is superfluous as well as one of the original nine coherence conditions. In this paper, they defined a transition between pseudomonads and proved that the transitions and liftings are essentially the same.
R.F.C. Walters in his doctoral dissertation [Walters, 1970], gave us a no-iteration presentation of a monad and its algebras and with these ideas of no-iteration F. Marmolejo and R. Wood rewrote the distributive laws for monads in [Marmolejo and Wood, 2010] and extended the notion of no-iteration monads and their algebras to higher dimensional categories in [Marmolejo and Wood, 2013].
The natural continuation for these ideas is to give a definition of a no-iteretion transition and a no-iteration distributive law between pseudomonads and prove that these are essentially the same as the usual definitions (this is done in my Ph.D. thsesis). In this talk we will cover these definitions and also the proofs.

References

[Beck, 1969] J. Beck. *Distributive laws.* Lecture Notes in Mathematics 80, Springer-Verlag, 1969. pp. 119-140.

[Walters, 1970] R.F.C. Walters. *A categorical approach to universal algebra.* Ph.D. Thesis. Australian National University, 1970.

[Marmolejo, 1999] F. Marmolejo. *Distributive laws for pseudomonads.* Theory and Applications of Categories, Vol. 5, No. 5, 1999, pp. 91-147.

[Marmolejo and Wood, 2008] F. Marmolejo and R.J. Wood. *Coherence for pseudodistributive laws revisited.* Theory and Applications of Categories, Vol. 20, No. 6, 2008, pp. 74-84.

[Marmolejo and Wood, 2010] F. Marmolejo and R.J. Wood. *Monads as extension systems - no iteration is necessary.* Theory and Applications of Categories, Vol. 24, No. 4, 2010, pp. 84-113.

[Marmolejo and Wood, 2013] F. Marmolejo and R.J. Wood. *No iteration pseudomonads.* Theory and Applications of Categories, Vol. 28, No. 14, 2013, pp. 371-402.

[Fiore, Gambino, Hyland and Winskel, 2017] M. Fiore, N. Gambino, M. Hyland and G. Winskel. *Relative pseudomonads, Kleisli bicategories and substitution monoidal structures.* Selecta Mathematica New Series, Vol. 24, 2018, pp. 2791-2830.

[Tanaka, 2005] Miki Tanaka. *Pseudo-Distributive Laws and a Unified Framework for Variable Binding.* Ph.D. Thesis. University of Edinburgh, 2005.

Non-cartesian internalisation and enriched quasi-categories

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Abstract.

The classical nerve functor $N : \mathrm{Cat} \hookrightarrow \mathrm{SSet}$ characterises categories as simplicial sets satisfying a unique lifting condition. Relaxing the uniqueness leads to the definition of a quasi-category as a simplicial set satisfying the weak Kan condition. If one wants to consider enriched quasi-categories, a natural question to ask is the following: Given a suitable monoidal category $(\mathcal{V}, \otimes, I)$, what is the nerve of a $\mathcal{V}$-enriched category? One might expect this to result in a simplicial object in $\mathcal{V}$, but a quick observation will show that no reasonable functor landing in simplicial objects $\mathcal{V}\,\mathrm{Cat} \to S\mathcal{V}$ exists when $\mathcal{V}$ is not cartesian monoidal (e.g. when $\mathcal{V}$ the category of vector spaces with the tensor product).

In this talk, I will present *tensor-simplicial* or *templicial objects* $S_{\otimes}\mathcal{V}$ as a generalisation of simplicial sets in the non-cartesian context. This is joint work with Wendy Lowen [11], and it fits the following scheme:

$$\begin{tikzcd}
\text{Categories internal to } \mathcal{V} \arrow[r, phantom, "\underset{\text{if } \otimes = \times}{\simeq}"] \arrow[d, hook, "N_{\mathcal{V}}"'] & \text{Categories internal to } (\mathcal{V}, \otimes, I) \arrow[r, phantom, "\supset"] \arrow[d, hook, "N_{\mathcal{V}}"'] & \mathcal{V}\text{-enriched categories} \arrow[d, hook, "N_{\mathcal{V}}"'] \\
\text{Simplicial objects in } \mathcal{V} \arrow[r, phantom, "\underset{\text{if } \otimes = \times}{\simeq}"] & \text{Colax monoidal functors } \mathbf{\Delta}_f^{op} \to \mathcal{V} \arrow[r, phantom, "\supset"] & \text{Templicial objects in } \mathcal{V}
\end{tikzcd}$$

where each $N_{\mathcal{V}}$ is a fully faithful right-adjoint generalising N. Categories internal to a (not necessarily cartesian) monoidal category $(\mathcal{V}, \otimes, I)$ were introduced by Aguiar in [1]. They recover classical internal categories when $\mathcal{V}$ is cartesian monoidal, and contain $\mathcal{V}$-enriched categories as a subclass. This picture extends nicely to higher dimensions. Let $\mathbf{\Delta}_f \subseteq \mathbf{\Delta}$ be the monoidal category of finite intervals, then a colax monoidal functor $\mathbf{\Delta}_f^{op} \to \mathcal{V}$ may be considered as a "simplicial object internal to $(\mathcal{V}, \otimes, I)$". It was already observed by Leinster [10] that these precisely recover $S\mathcal{V}$ when $\mathcal{V}$ is cartesian.

We posit templicial objects as a suitable context to define an enriched variant of Joyal's quasi-categories [8]. In particular this is motivated by (noncommutative) algebraic geometry - where the strictly enriched model of dg-categories play a central role - and algebraic deformation theory [3]. Our main results are the following, some of which I will outline during the talk:

1. In [11], we identified an analogue of the weak Kan condition for templicial objects. If it is satisfied we call the templicial object a *quasi-category in* $\mathcal{V}$, which precisely recovers classical quasi-categories when $\mathcal{V} = \mathrm{Set}$. To express this condition, we make essential use of the necklaces of [2][6]. In a separate project with Violeta Borges Marques [4], we construct a Reedy structure on necklaces.

2. Employing necklaces, we construct a general framework for producing enriched variants of other nerves as well, such as the homotopy-coherent nerve [5], the dg-nerve [13] and the cubical nerve

[9], all of which land in $S_\otimes\mathcal{V}$. This moreover allows to obtain explicit descriptions of their left-adjoints.

3. Let k be a commutative ring. Through the enriched variant of the dg-nerve, we show in [12] an equivalence of categories between non-negatively graded dg-categories over k and quasi-categories in k-modules equipped with a certain Frobenius structure.

4. Our current main goal is to construct a model structure on the category of templicial objects - analogous to Joyal's model structure on simplicial sets - such that the enriched homotopy coherent-nerve $N^{hc}_{\mathcal{V}} : S\mathcal{V}\text{-Cat} \to S_\otimes\mathcal{V}$ becomes a Quillen equivalence. This would establish quasi-categories in $\mathcal{V}$ as a model for $S\mathcal{V}$-enriched ∞-categories in the sense of [7]. Moreover, we expect templicial objects to be model monoidal, which fails for $S\mathcal{V}$-Cat.

   I will outline some results in this direction. For instance, when $\mathcal{V} = \mathrm{Vect}(k)$, templicial vector spaces define a category of cofibrant objects with a compatible symmetric monoidal structure.

References

[1] M. Aguiar, *Internal categories and quantum groups*, ProQuest LLC, Ann Arbor, M, 1997.

[2] H.J. Baues, *Geometry of loop spaces and the cobar construction*, Mem. Amer. Math. Soc., 25 (1980), ix+171.

[3] V. Borges Marques, W. Lowen and A. Mertens, Deformations of quasi-categories in modules, preprint arXiv:2308.09148, 2023.

[4] V. Borges Marques and A. Mertens, *The category of necklaces is Reedy monoidal*, Theory Appl. Categ., 41 (2024), no. 3, 71–85.

[5] J.-M. Cordier, *Sur la notion de diagramme homotopiquement cohérent*, Cahiers de Topologie et Géom. Différentielle Catégoriques, 23 (1982), no. 1, 93–112.

[6] D. Dugger and D Spivak *Rigidification of quasi-categories*, Algebr. Geom. Topol., 11 (2011), no. 1, 225–261.

[7] D. Gepner and R. Haugseng, *Enriched ∞-categories via non-symmetric∞-operads, Adv. Math.*, 279 (2015), 575–716.

[8] A. Joyal, The theory of quasi-categories and its applications. *unpublished*, 2008.

[9] B. Le Grignou, *Cubical model categories and quasi-categories*, Theory Appl. Categ., 35 (2020), no. 18, 622–664.

[10] T. Leinster, Homotopy algebras for operads, preprint arXiv:math/0002180v1, 2000.

[11] W. Lowen and A. Mertens, *Enriched quasi-categories and the templicial homotopy coherent nerve*, Alg. Geom. Topol., accepted for publication, 2024.

[12] W. Lowen and A. Mertens, Frobenius templicial modules and the dg-nerve, preprint, arXiv:2005.04778v3, 2023.

[13] Lurie, J. Higher algebra. *unpublished*, 2017.

Aspects of 2-dimensional Elementary Topos Theory

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Abstract.

We will present the main results of our PhD thesis, that contribute to expand 2-dimensional elementary topos theory. As elementary topos theory has had an enormous success, with numerous applications to geometry and logic, we believe it is very fruitful to generalize the theory to dimension 2. Introduced by Weber in [5], 2-dimensional elementary topos theory is still at its beginning, but with a high potential of application to stacks, classifying topoi and 2-categorical logic. Lawvere's idea of an elementary topos was that of a generalized universe of sets. On this line, an elementary 2-topos is a generalized universe of categories.

We will focus in particular on the concept of 2-classifier, which was introduced by Weber in [5] and is the 2-categorical generalization of the notion of subobject classifier. The idea is to classify discrete opfibrations, that have higher dimensional fibres. And the archetypal example is given by the category of elements construction, exhibiting $\mathcal{C}at$ as the archetypal elementary 2-topos. So, interestingly, a 2-classifier can also be thought of as a Grothendieck construction inside a 2-category. We will introduce a notion of good 2-classifier, that captures well-behaved 2-classifiers and is closer to the point of view of logic. The idea is to still have as classifier an object of generalized truth values together with the choice of a verum, as in dimension 1. This is realized by upgrading the classification process from one regulated by pullbacks (in dimension 1) to one regulated by comma objects (in dimension 2).

We will present a novel technique of reduction of the study of 2-classifiers to dense generators, developed in our [4]. Dense generators capture the idea of a family of objects that generate all the other ones via nice colimits; the preeminent example is given by representables in categories of presheaves. We will show that both the conditions of 2-classifier and what gets classified by a 2-classifier can be checked just over the objects that form a dense generator. This substantially reduces the work needed to prove that something is a 2-classifier. For example, applied to the archetypal case of $\mathcal{C}at$, it allows us to deduce all the major properties of the category of elements construction from the trivial observation that everything works well over the singleton category.

We will then apply the theorems of reduction of 2-classifiers to dense generators to produce a good 2-classifier in stacks, classifying all discrete opfibrations with small fibres. This generalizes to dimension 2 the fundamental result that Grothendieck topoi are elementary topoi. Indeed, Grothendieck topoi are given by categories of sheaves, and stacks are precisely the 2-categorical generalization of sheaves. Stacks still capture the idea to glue together compatible local data into a global datum, but the compatibility conditions are only required up to isomorphism. They are a key object of study of the modern algebraic geometry, and they have solved numerous problems (e.g. moduli problems) that were not solvable using ordinary spaces or 1-dimensional sheaves. Thanks to our results, the 2-categories of stacks, i.e. Grothendieck 2-topoi, will be elementary 2-topoi.

To reach our good 2-classifier in stacks, we will first apply our theorems of reduction of 2-classifiers to dense generators to produce a good 2-classifier in prestacks (i.e. 2-presheaves). We will achieve this by using an indexed version of the Grothendieck construction, developed in our joint work with Caviglia [1]. This gives a pseudonatural equivalence of categories between opfibrations over a fixed base in the 2-category of 2-copresheaves and 2-copresheaves on the Grothendieck construction of the fixed base. Our result can be interpreted as the result that every (op)fibrational slice of a Grothendieck 2-topos is a Grothendieck 2-topos. So this generalizes to dimension 2 the so-called fundamental theorem of elementary topos theory, in the Grothendieck topoi case. Our good 2-classifier in prestacks involves a 2-dimensional generalization of the concept of sieve, that is a key element of the notion of Grothendieck topology. We will then restrict our good 2-classifier in prestacks to one in stacks, via factorization arguments and our theorems of reduction to dense generators. In particular, we will generalize closedness of a sieve to dimension 2. Our results also solve a problem posed by Hofmann and Streicher in [2] when attempting to lift Grothendieck universes to sheaves.

The driving idea behind our technique of reduction to dense generators is to express an arbitrary object as a nice colimit of the dense generators and induce the required data using the universal property of the colimit. In order to handle such colimits in our 2-categorical setting, we apply the calculus of colimits in 2-dimensional slices that we developed in [3]. In particular, our calculus generalizes to dimension 2 the well-known fact that a colimit in a 1-dimensional slice category is precisely the map from the colimit of the domains of the diagram that is induced by the universal property. We show that the appropriate 2-dimensional slice to consider for this is the lax slice, and that the appropriate 2-dimensional colimits to consider are marked conical colimits.

References

[1] E. Caviglia and L. Mesiti, Indexed Grothendieck construction, preprint arXiv:2307.16076, 2023.

[2] M. Hofmann and T. Streicher, Lifting Grothendieck universes, preprint `https://www2.mathematik.tu-darmstadt.de/~streicher/NOTES/lift.pdf`, 1997.

[3] L. Mesiti, Colimits in 2-dimensional slices, preprint arXiv:2305.01494, 2023.

[4] L. Mesiti, 2-classifiers via dense generators and Hofmann-Streicher universe in stacks, preprint arXiv:2401.16900, 2024.

[5] M. Weber, *Yoneda structures from 2-toposes*, Appl. Categ. Struct., 15(3):259–323, 2007.

Eilenberg-Moore Bicategories for Opmonoidal Pseudomonads

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Abstract. Given an monad T on a strict monoidal category $(\mathcal{V}, \otimes, I)$, suitably coherent natural transformations with components of the form $\chi_{X,Y} : T(X \otimes Y) \to TX \otimes TY$ and $\iota : TI \to I$ comprise a structure known as an opmonoidal monad. In the presence of such structure, the category of Eilenberg-Moore algebras $\mathcal{V}^T$ for the underlying monad also inherits a monoidal structure [6, 8]. Similarly, braidings and symmetries also lift to categories of algebras under suitable compatibility conditions. These results follow from two-dimensional monad theory, specifically the theory of Eilenberg-Moore objects in 2-categories of strict algebras and lax morphisms [4]. Alternatively, they also follow from the observation that the Eilenberg-Moore construction and products are both limits, and hence commute with one another, and opmonoidal monads are monoids in the 2-category of monads [10, 11].

In this talk I will discuss how these results extend to the two-dimensional setting. In this setting, the Eilenberg-Moore construction for pseudomonads is still a limit [3], however the theory of limits for lax morphisms of algebras for three-dimensional monads is far more complicated and not as well-developed [9]. Moreover, the appropriate monoidal structures on 2-categories [1] are now monoids in a non-cartesian monoidal structure, and as such monoidality of the Eilenberg-Moore construction for pseudomonads needs to be checked directly. Indeed, an important stepping stone is to check that the **Gray**-tensor product actually extends to pseudomonads. Once we have done all of this we find that the 2-category of pseudoalgebras $\mathcal{V}^T$ [5] inherits a monoidal structure that is slightly weaker than the original structure on $\mathcal{V}$, with associativity and unit laws holding up to 2-natural isomorphisms which satisfy the usual monoidal category axioms on the nose. We also describe similar liftings to pseudoalgebras for braidings, syllapses and symmetries that are suitably compatible with the pseudomonad structure.

This talk is based on results in [7]. Motivating applications include two-dimensional linear algebra and bicategorical models of linear logic [2]. This research is supported by EPSRC under grant EP/V002325/2.

References

[1] B. Day, R. Street, *Monoidal bicategories and Hopf algebroids*, Advances in Mathematics, Vol. 129, Issue 1, pp. 99-157, (1997)

[2] M. Fiore, N. Gambino, M. Hyland, Monoidal Bicategories, Differential Linear Logic, and Analytic Functors, *In preparation*

[3] S. Lack, *A coherent approach to pseudomonads*, Advances in Mathematics, Vol. 152, pp. 179-202, (2000)

[4] S. Lack, *Limits for Lax Morphisms*, Applied Categorical Structures, Vol. 13, pp. 189–203 (2005).

[5] F. Marmolejo, *Doctrines whose structure forms a fully faithful adjoint string*, Theory and Applications of Categories, Vol. 3, No. 2, pp. 22–23, (1997).

[6] P. McCrudden, *Opmonoidal monads*, Theory and Applications of Categories, Vol. 10, No.19, pp.469–485, (2002).

[7] A. Miranda, Eilenberg-Moore Bicategories for Opmonoidal Pseudomonads, preprint `https://arxiv.org/abs/2402.11703`.

[8] I. Moerdijk, *Monads on tensor categories*, Journal of Pure and Applied Algebra, Vol. 168, Issues 2–3, pp. 189-208, (2002)

[9] A. J. Power, *Three-dimensional monad theory*, Categories in algebra, geometry and mathematical physics, Vol 431, pp 405-426, (2007)

[10] R. Street, *The formal theory of monads*, Journal of Pure and Applied Algebra, Vol. 2, pp. 149-168, (1978)

[11] M. Zawadowski, *The Formal Theory of Monoidal Monads*, Journal of Pure and Applied Algebra, Vol. 216, pp. 1932-1942, (2012)

Finitary semantics and languages of λ-terms

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Abstract.

This is joint work with Sam van Gool, Paul-André Melliès and Tito Nguyễn.

There is a growing connection between automata theory and the theory of λ-calculus. Indeed, the Church encoding shows that finite words and ranked trees are simply typed λ-terms. For instance, words over the alphabet $\Sigma = \{a, b\}$ correspond to λ-terms of type

$$\mathsf{Church}_\Sigma \quad := \quad \underbrace{(\mathtt{o} \Rightarrow \mathtt{o})}_{a \text{ transition}} \Rightarrow \underbrace{(\mathtt{o} \Rightarrow \mathtt{o})}_{b \text{ transition}} \Rightarrow \underbrace{\mathtt{o}}_{\text{initial state}} \Rightarrow \underbrace{\mathtt{o}}_{\text{output state}}$$

Moreover, their semantic interpretations in the cartesian closed category **FinSet** coincides with their behavior in finite deterministic automata. This semantic observation led Salvati to define the notion of **recognizable language** in [7] as any set of λ-terms of a given type A of the form

$$\{M \in \Lambda(A) \mid [\![M]\!]_Q \in F\} \qquad \text{for some finite set } Q \text{ and subset } F \subseteq [\![A]\!]_Q.$$

The recognizable languages of type Church_Σ are then exactly the regular languages of words, seen through the Church encoding. Moreover, Salvati has shown that, for any type A, languages of λ-terms of that type assemble into a Boolean algebra. This definition, using finite sets, extends to any cartesian closed category.

There is another, more syntactic link between automata theory and λ-calculus. A seminal result by Hillberand and Kanellakis [3] states that a set of finite words is a regular language if and only if its characteristic function is λ-definable, modulo a type-casting operation sending any $M \in \Lambda(A)$ to $M[B] \in \Lambda(A[B])$. This observation is at the heart of the implicit automata program started in [5], which shows an analogous correspondence between star-free languages and planar λ-terms.

This line of work yields another, more syntactic notion of regular language of λ-terms of type A, implicit in the work of Hillebrand and Kanellakis. A **syntactically regular language** of λ-terms of a given type A is any set of the form

$$\{M \in \Lambda(A) \mid R\ M[B] =_{\beta\eta} \mathsf{true}\} \qquad \text{for some type } B \text{ and } \lambda\text{-term } R \in \Lambda(A[B] \Rightarrow \mathsf{Bool})$$

where Bool is the type $\mathtt{o} \Rightarrow \mathtt{o} \Rightarrow \mathtt{o}$ and true is the first projection.

In [4], we show that, for a large class of sufficiently well-behaved cartesian closed categories, the associated recognizable languages are exactly the syntactically regular ones. More precisely:

Theorem 1 (§7 of [4])**.** *A language of λ-terms of type A is recognizable by a non-thin well-pointed locally finite cartesian closed category if and only if it is syntactically regular.*

Theorem 1 provides evidence that the notion of recognizable language of λ-terms is robust, and does not depend on the category of finite sets. Its proof relies on a new construction on cartesian closed categories called **squeezing**, which is inspired by normalization by evaluation.

In [2], we have introduced profinite λ-terms, using semantic interpretation in finite sets, which assemble into a cartesian closed category **ProLam**. Profinite λ-terms of type Church_Σ are exactly the profinite words, and they extend the correspondance coming from Stone duality with regular languages [6, 1] in the following way:

Theorem 2 (Proposition 3.4 of [2])**.** *The space of profinite λ-terms of type A is the Stone dual of the Boolean algebra of regular languages of λ-terms of type A.*

Dually, the combination of Theorem 1 with Theorem 2 shows that the space of profinite λ-terms, initially defined in the setting of semantic interpretation in finite sets, does not depent on that construction.

References

[1] Mai Gehrke, Serge Grigorieff, and Jean-Éric Pin. Duality and equational theory of regular languages. In Automata, Languages and Programming, 35th International Colloquium, ICALP 2008, Reykjavik, Iceland, July 7-11, 2008, Proceedings, volume 5126 of Lecture Notes in Computer Science, pages 246–257. Springer, 2008. doi:10.1007/978-3-540-70583-3_21.

[2] Sam van Gool, Paul-André Melliès, and Vincent Moreau. Profinite lambda-terms and parametricity. Electronic Notes in Theoretical Informatics and Computer Science, Volume 3 - Proceedings of MFPS XXXIX , November 2023. URL: https://entics.episciences.org/12280, doi:10.46298/entics.12280.

[3] Gerd G. Hillebrand and Paris C. Kanellakis. On the expressive power of simply typed and let-polymorphic lambda calculi. In Proceedings, 11th Annual IEEE Symposium on Logic in Computer Science, New Brunswick, New Jersey, USA, July 27-30, 1996, pages 253–263. IEEE Computer Society, 1996. doi:10.1109/LICS.1996.561337.

[4] Vincent Moreau and Lê Thành Dũng (Tito) Nguyễn. Syntactically and semantically regular languages of lambda-terms coincide through logical relations. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2024. URL: https://drops.dagstuhl.de/entities/document/10.4230/LIPIcs.CSL.2024.40, doi:10.4230/LIPICS.CSL.2024.40.

[5] Lê Thành Dũng Nguyễn and Cécilia Pradic. Implicit automata in typed λ-calculi I: aperiodicity in a non-commutative logic. In 47th International Colloquium on Automata, Languages, and Programming, ICALP 2020, July 8-11, 2020, Saarbrücken, Germany, volume 168 of LIPIcs, pages 135:1–135:20. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020. doi:10.4230/LIPIcs.ICALP.2020.135.

[6] Nicholas Pippenger. Regular languages and stone duality. Theory of Computing Systems, 30(2):121–134, April 1997. URL: http://dx.doi.org/10.1007/BF02679444, doi:10.1007/bf02679444.

[7] Sylvain Salvati. Recognizability in the simply typed lambda-calculus. In Logic, Language, Information and Computation, 16th International Workshop, WoLLIC 2009, Tokyo, Japan, June 21-24, 2009. Proceedings, volume 5514 of Lecture Notes in Computer Science, pages 48–60. Springer, 2009. doi:10.1007/978-3-642-02261-6_5

Toposes as Lex-Presentable Categories

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Abstract.

When learning topos theory, one encounters a bewildering array of *exactness conditions* — special interactions between (weighted) colimits and finite limits which obtain in the category of sets, but not in general categories. These are all of generally the same form, but they differ enough in the details to challenge the memory. Categories with some of these exactness conditions (regular, coherent, geometric, exact, lextensive, etc.̃) play an important role in topos theory, and these sorts of categories correspond to doctrines of geometric logic or type theory.

In their beautiful paper "Lex colimits", Garner and Lack smooth this story out by showing that these exactness conditions emerge by co-completing not in the 2-category **Cat** of categories, but instead relative to KZ-doctrines on the 2-category **Lex** of categories with finite limits and finite limit preserving functors between them. They are therefore able to reframe Giraud's characterization of sheaf toposes in the following way: a topos is a lex-cocomplete category which is (locally) presentable. However, while the notion of lex-cocompletion takes place in **Lex**, the notion of presentability is the original notion which takes place in **Cat**. Can we give a characterization of toposes as lex-presentable categories taking place fully in **Lex**?

In their delightful paper "Accessibility and presentability in 2-categories", Di Liberti and Loregian define presentability relative to a *KZ-context* $\nu : \mathcal{S} \hookrightarrow \mathcal{P}$ (a fully faithful inclusion of KZ-doctrines) on an arbitrary 2-category. When $\mathcal{S}$ has a complementary KZ-context $\mathcal{D} \hookrightarrow \mathcal{P}$ so that the induced map $\mathcal{S}\mathcal{D} \to \mathcal{P}$ is an equivalence (and under a few other minor assumptions), Di Liberti and Loregian provide a Gabriel-Ulmer duality between *petite* $\mathcal{D}$-cocomplete objects and $\mathcal{S}$-presentable objects in this abstract setting.

In this talk, we will apply Di Liberti and Loregian's theory to the 2-category **Lex** and the KZ-doctrine $\mathcal{P}$ of free cocompletion, interpreting the results using Garner and Lack's theory of lex colimits to see lex-κ-presentable categories as κ-coherent toposes. We'll begin by defining a notion of weighted colimit relative to a KZ-doctrine, which will bring Di Liberti and Loregian's abstract setting closer to Garner and Lack's formulation of lex cocompletion using weighted colimits. This will also allow us to describe taking lex colimits "one-by-one" and not just the operation of lex-cocompletion under a class. We will then discuss the conditions under which two classes of weights $\mathcal{S}$ and $\mathcal{D}$ (to equivocate between the classes and their free cocompletion KZ-doctrines, for the moment) are *complementary* and satisfy the assumptions necessary to apply Di Liberti and Loregian's Gabriel-Ulmer duality to produce a duality between $\mathcal{D}$-theories and $\mathcal{S}$-presentable toposes.

In the proof of their Theorem 6.4, Garner and Lack define a sub-canonical topology $j_{\mathcal{D}}$ on $\mathcal{D}$-cocomplete categories C for which the Yoneda embedding $y : C \to \mathbf{Sh}(C, j_{\mathcal{D}})$ is $\mathcal{D}$-cocontinuous. We will say that $\mathcal{S}$ and $\mathcal{D}$ are *complementary classes of weights* when (1) any colimit may be expressed

as an $\mathcal{S}$-weighted colimit of $\mathcal{D}$-weighted colimits and (2) a $\mathcal{S}$-weighted colimit (in presheaves) of $j_{\mathcal{D}}$-sheaves is still a $j_{\mathcal{D}}$-sheaf. We'll see that complementary classes of colimits satisfy Di Liberti and Loregian's Assumptions 3.14 and therefore admit a Gabriel-Ulmer duality.

We'll conclude by observing that κ-filtered colimits ($\mathcal{S}$) are complementary to κ-small colimits ($\mathcal{D}$), so that Di Liberti-Loregian's Gabriel-Ulmer duality gives us the familiar dual equivalence between κ-exact categories (κ-complete pretoposes)[1] with lex κ-cocontinuous functors between them and κ-coherent toposes with relatively κ-tidy geometric morphisms between them.[2][3]

References

[1] R. Garner, and S. Lack, *Lex colimits*, Journal of Pure and Applied Algebra 216 (2012), no. 6, 1372–1396.

[2] I. Di Liberti, and F. Loregian *Accessibility and presentability in 2-categories*, Journal of Pure and Applied Algebra 227 (2023), no. 1.

[3] I. Di Liberti; Ramos González, Julia. Gabriel-Ulmer duality for topoi and its relation with site presentations. Appl. Categ. Structures 28 (2020), no. 6, 935–962.

[1] This is true for $\kappa \geq \omega$; for $\kappa = 1$, $\mathcal{D}$ is the identity and $\mathcal{S} = \mathcal{P}$ is free cocompletion and the duality is between small lex categories and free toposes.

[2] I believe that this duality between pretoposes and coherent toposes was first observed by Makkai, but I could not find a reference.

[3] We note that this Gabriel-Ulmer duality differs slightly from that of [3] which begins with ordinary Gabriel-Ulmer duality and carves it down to toposes. They establish a duality with *proto-toposes* rather than pretoposes.

Double categories of relations relative to factorization systems

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Abstract.

Relations and spans in a category have been studied in the context of double categories. The recent work [1] investigated a characterization of the double category of spans in a finitely complete category, and later in [4], the double category of relations in a regular category was characterized. In this talk, we will present a generalization of these results by introducing *double categories of relations relative to stable orthogonal factorization systems*.

Characterization of double categories of relations. Our presentation will be based on the notion of relations relative to a factorization system (E, M) on a category $\mathbf{C}$, or M-relations, which are defined as spans jointly belonging to the right class M. Taking arrows as tight arrows (vertical arrows) and M-relations as loose arrows (horizontal arrows), we obtain the double category of M-relations. For a double category $\mathbb{D}$ with a terminal object, we can define a class $\mathrm{Fib}(\mathbb{D})$ of tight arrows called *fibrations*, which in the case of the double category of profunctors, are precisely the discrete fibrations. We will explain the conditions on a double category $\mathbb{D}$ under which the class of fibrations $\mathrm{Fib}(\mathbb{D})$ becomes the right class of a stable orthogonal factorization system (E, M) on the category of tight arrows, and the double category $\mathbb{D}$ is equivalent to the double category of M-relations.

Layering classes of factorization systems via double categorical properties. We will also discuss some important properties of factorization systems that are reflected in the double categorical viewpoint. Our characterization theorem translates properties of factorization systems into properties of double categories, and vice versa, as shown in Table 1. The property of *unit-pureness* on double categories will be explored, which is the counterpart of the property of a factorization system having the left class included in the class of epimorphisms. By the correspondence, we reprove the characterization of the double category of spans and that of relations in a regular category from the unified viewpoint, providing a rationale that some conditions, such as unit-pureness and local posetality, are essential in the original proofs.

Cauchy condition on the double category of M**-relations.** Another property of double categories we will discuss is the Cauchy condition. A category is Cauchy complete if any left adjoint profunctor from (or into, depending on the convention) it is representable. This leads to the definition of Cauchy double categories ([5]) that all loose adjoints in it are representable by tight arrows. We will show that in a unit-pure double category of M-relations, left adjoint loose arrows are "ana-tight-arrows", meaning that they are of the form

$$A \xleftarrow{m} B \xrightarrow{f} C$$

where m is a monomorphism belonging to the left class E of the factorization system. It brings us to the characterization of unit-pure Cauchy double categories of relations as those whose right class M of the corresponding factorization system includes all monomorphisms. As previously noted in [6], the Cauchy condition is the categorical formulation of the unique-choice principle.

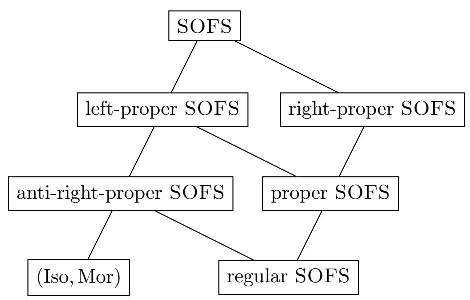
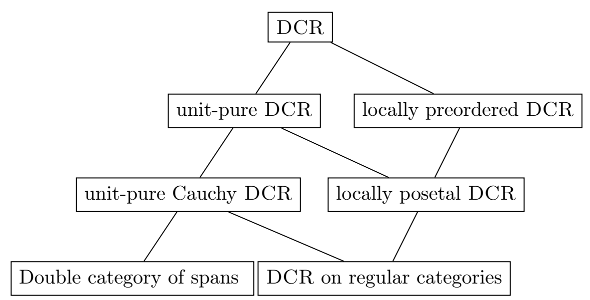
SOFSs on finitely complete categories	Double categories of relations (DCRs)
	

Table 1: Correspondence between classes of stable orthogonal factorization systems and double categories of relations.

References

[1] Evangelia Aleiferi. *Cartesian double categories with an emphasis on characterizing spans.* PhD thesis, Dalhousie University, Halifax, Nova Scotia, 2018.

[2] Keisuke Hoshino and Hayato Nasu. Double categories of relations relative to factorisation systems. preprint, arXiv:2310.19428, 2024.

[3] G. M. Kelly. A note on relations relative to a factorization system. In *Category theory (Como, 1990)*, volume 1488 of *Lecture Notes in Math.*, pages 249–261. Springer, Berlin, 1991.

[4] Michael Lambert. Double categories of relations. *Theory Appl. Categ.*, 38:Paper No. 33, 1249–1283, 2022.

[5] Robert Paré. Morphisms of rings. In *Joachim Lambek: the interplay of mathematics, logic, and linguistics*, volume 20 of *Outst. Contrib. Log.*, pages 271–298. Springer, Cham, 2021.

[6] Giuseppe Rosolini. A note on Cauchy completeness for preorders. *Riv. Mat. Univ. Parma (6)*, 2*:89–99, 1999.

Coalgebraic enrichment of categorical W-types

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Abstract.
In both the traditions of functional programming and categorical logic, one takes the perspective that most data types should be obtained as initial algebras of polynomial endofunctors. For instance, the natural numbers are obtained as the initial algebra of the endofunctor $X \mapsto X + 1$, assuming that the category in question (often the category of sets) has a terminal object 1 and a coproduct $+$. Much theory has been developed around this approach, which culminates in the notion of W-types [2, 3].

In another tradition, that of categorical algebra, algebras (in the traditional sense) over a field k are studied. It has been long understood (going back at least to Wraith and Sweedler, according to [1]) that the category of k-algebras is naturally enriched over the category of k-coalgebras, a fact which has admitted generalization to several other settings (e.g. [1, 5]). Here, we generalize those classic results to the setting of an endofunctor on a category, and in particular those endofunctors that are considered in the theory of W-types.

That is to say, this work is the beginning of a development of an analogue of the theory of W-types – not based on the notion of initial objects in a *category* of algebras, but rather on a generalized notion of initial object in a *coalgebra enriched category* of algebras. The hom-coalgebras of our enriched category carry more information than the hom-sets in the unenriched category that is usually considered in the theory of W-types. We are then able to generalize the notion of *initial algebra*, taking inspiration from the theory of weighted limits, which is more expressive, and thus can be used to specify more objects than the usual notion of initial algebra. Because of our move to the enriched setting, then, we have better control than in the unenriched setting, and we are able to specify more data types than just those which are captured by the theory of W-types.

Our main theorem is the following.

Theorem. *Let $(\mathsf{C}, \otimes, \mathbb{I}, \underline{\mathsf{C}}(-,-))$ be a locally presentable symmetric monoidal closed category. Let $F : \mathsf{C} \to \mathsf{C}$ be an accessible lax symmetric monoidal endofunctor. Then the category Alg_F of F-algebras is enriched, tensored, and powered over the symmetric monoidal category CoAlg_F of F-coalgebras.*

We show that many endofunctors of interest in the theory of W-types satisfy these hypotheses. For instance, Set is a locally presentable symmetric monoidal closed category. The following functors

on a locally presentable symmetric monoidal closed category satisfy the hypotheses: the identity functor, any constant functor at a commutative monoid, the coproduct of two functors that satisfy the hypothesis, and the product two functors that satisfy the hypotheses.

In particular, the functor $X \mapsto X + 1$ on Set satisfies the hypotheses, and we work out very explicitly what the enrichment (and tensoring and cotensoring) tells in this situation. In this concrete case, we see that the enrichment encodes a notion of *partial algebra homomorphism*, whereas the usual category of algebras encodes the notion of *total algebra homomorphism*.

We then observe that there is an implicit parameter in the notion of initial algebra which we may now vary. One might think of an initial object as a certain *colimit*, but in reality, an initial object in a category C is usually (equivalently) defined as an object I with the property that $\mathsf{hom}(I, X) = \{*\}$ for every $X \in \mathsf{C}$. That is, I is the vertex of a cone over the identity functor on C with the special property that each leg of the cone (at an object $X \in \mathsf{C}$) is the only morphism of $\mathsf{hom}(I, X)$. The reader might know that as such, an initial object can always be defined as the *limit* of the identity functor on C. Now that we are in the enriched setting, however, the appropriate notion of limit becomes that of *weighted limit* in which we are able ask not just that $\mathsf{hom}(I, X) = \{*\}$ but that $\mathsf{hom}(I, X) = W$ for any object W. Thus, we make the following definition.

Definition. *Consider a monoidal category* $(\mathsf{C}, \otimes, \mathbb{I}, \underline{\mathsf{C}}(-,-))$ *and endofunctor* $F : \mathsf{C} \to \mathsf{C}$ *satisfying the hypotheses of the above theorem.*

For $W \in \mathsf{CoAlg}_F$*, we define the* W*-initial algebra to be the limit of the identity functor on* Alg_F *(viewed as the enriched categories described in the above theorem) weighted by the constant functor* $\mathsf{Alg}_F \to \mathsf{CoAlg}_F$ *at* W*.*

With this, we are able to obtain algebras that represent, for instance in the example of the functor $X \mapsto X + 1$, partial induction.

Our hope is that all of this extra structure discovered in the quite classical categorical semantics of functional programming languages can be used to augment them.

This talk reports on [4] and further work in progress.

References

[1] Martin Hyland, Ignacio López Franco, and Christina Vasilakopoulou. Hopf measuring comonoids and enrichment. *Proc. Lond. Math. Soc. (3),* 115(5):1118–1148, 2017.

[2] Per Martin-Löf. Intuitionistic type theory. *Naples: Bibliopolis,* 1984

[3] Ieke Moerdijk and Erik Palmgren. Wellfounded trees in categories. Annals of Pure and Applied Logic, 104(1):189–218, 2000.

[4] Paige Randall North and Maximilien Péroux. Coinductive Control of Inductive Data Types. In *10th Conference on Algebra and Coalgebra in Computer Science (CALCO 2023). Leibniz International Proceedings in Informatics (LIPIcs)*, Volume 270, pp. 15:1-15:17, Schloss Dagstuhl – Leibniz-Zentrum für Informatik (2023).

[5] Maximilien Péroux. The coalgebraic enrichment of algebras in higher categories. *Journal of Pure and Applied Algebra,* 226(3):106849, 2022.

On a (terminally connected, pro-etale) factorization system for geometric morphisms

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Abstract.

A classical result of topos theory states that any locally connected geometric morphism $f : \mathcal{F} \to \mathcal{E}$ factorizes uniquely as a connected geometric morphism followed by an etale geometric morphism, where the etale part is presented by the essential image $f_!(1_{\mathcal{F}})$ of the terminal object – morally, the object of connected components of the image of f. Such a factorization still makes sense for the wider class of essential geometric morphisms, though the left part of this factorization is no longer connected; however it still is *terminally connected*.

Terminally connected geometric morphisms are in some sense those which are only connected "from the point of view of the terminal object". In the essential world, this condition (as it was first introduced in [1]) says that the essential image preserves the terminal object; but a reformulation of this condition, which makes sense for arbitrary geometric morphisms, is that there exists a natural isomorphism $\mathcal{F}[1_{\mathcal{F}}, f^*(-)] \simeq \mathcal{E}[1_{\mathcal{E}}, -]$, or in words, that the inverse image uniquely lifts global elements. Terminally connected geometric morphisms have some interesting stability properties along Beck-Chevalley squares or also left to bicomma squares (though not along bipullback); moreover one can show that they are exactly those that are left orthogonal to etale geometric morphisms, which suggests they form the left class of a factorization system for all geometric morphisms.

This requires first to identify a correct generalization of etale morphisms on the right. Indeed, in full generality a geometric morphism may lack an essential image part displaying connected components; yet its inverse image part, as a lex functor, nevertheless possesses a *left pro-adjoint*, that is, a relative left adjoint along its free completion under cofiltered limits. In some sense, though the connected components of an arbitrary geometric morphism may not be indexed by a discrete set internal to the codomain topos, they will nevertheless form a pro-discrete internal locale. This suggests that a correct replacement of etale morphisms could be provided by those that are cofiltered limits of etale geometric morphisms, a.k.a *pro-etale geometric morphisms*.

Factorizing a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ through the etale geometric morphism $\mathcal{E}/E \to \mathcal{E}$ at a given object E of $\mathcal{E}$ amounts to providing a global element $a : 1_{\mathcal{F}} \to f^*E$; similarly factorizations through a pro-etale geometric morphism correspond to cofiltered diagrams in the category of elements $1_{\mathcal{F}} \downarrow f^*$. In particular there is a best such factorization through the pro-etale indexed by the cofiltered category of all global elements of the inverse image part; though this category is large, this cofiltered limit is always well-defined for $1_{\mathcal{F}} \downarrow f^*$ can be shown to admits a small initial subcategory thanks to

an accessibility argument. In particular any pro-etale geometric morphism can be reindexed by the category of elements of its own inverse image, which provides a canonical presentation.

If now one factorizes a geometric morphism through the pro-etale morphism indexed by all the possible global elements of the inverse image

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "f"] \arrow[dr, "t_f"'] & & \mathcal{E} \\ & \underset{(E,a) \in 1_{\mathcal{F}} \downarrow f^*}{\text{bilim}} \mathcal{E}/E \arrow[ur, "p_f"'] & \end{tikzcd}$$

the residual left part t_f is then always terminally connected, for in some sense all global elements are displayed in a faithful way in the cofiltered limit, a fact that can be proven concretely thanks to a formula presenting the bilimit topos as the sheaf topos over the pseudocolimits of etale sites. This begets an orthogonal *(terminally connected, pro-etale)* factorization for all geometric morphisms. A peculiar manifestation of this factorization was already known since [2] as the *Grothendieck-Verdier localization* at a point of a topos, the germ at this point.

A remarkable property of this factorization is that it allows for a canonical factorizations of lax 2-cells. In general 2-dimensional factorization systems do not come with a canonical way to relate the factorizations of two 1-cells related by a 2-cell; equivalently, the orthogonality condition for (pseudo)squares does not extend to either lax or oplax squares. But in this very case, it happens to be so: terminally connected morphisms enjoy a special lax orthogonality condition relative to pro-etale, which comes also with special cocomma stability property and factorization of 2-cells. This property is somewhat reminiscent of the comprehensive factorization (*initial, discrete opfibrations*) on **Cat** and we will see this is not a coincidence.

This talk will describe properties of terminally connected and pro-etale geometric morphisms, and give a throughout proof of the existence of their associated factorization. We will also give a more intrinsic characterization of pro-etale geometric morphisms as those whose inverse image generates the domain topos through fibers of global elements. We will also give a closely related factorization for locales morphisms and discuss their behavior along the localic reflection. We also discuss some syntax-semantics aspects and the relation with another factorization, the (*focalisation*, *terminally connected*) in **Lex** that is implicitly involved in Gabriel-Ulmer duality. We finally discuss the relation with another possible generalization of the (connected, etale) factorization, the *(connected, algebraic)* factorization identified in [3].

References

[1] O. Caramello, *Denseness conditions, morphisms and equivalences of toposes*, preprint arXiv:1906.08737, 2020.

[2] P.T. Johnstone, I. Moerdijk, *Local maps of toposes*, Proceedings of the London Mathematical Society, vol. s3-58, 1989.

[3] J. Lurie, *Higher topos theory*, Princeton University Press, 2009.

When limits are limits: Topological enrichment with an application to probability

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Abstract.
Sequential limits and colimits are often used, intuitively, to "approximate spaces" from below or from above. For example, the diagram in Set formed by finite sets and inclusion maps

$$\{0\} \hookrightarrow \{0,1\} \hookrightarrow \{0,1,2\} \hookrightarrow \{0,1,2,3\} \hookrightarrow \dots$$

has as colimit the set of natural numbers $\mathbb{N}$. The same is true more generally for filtered colimits and cofiltered limits. These examples are probably the motivation for the name "limit", by analogy with topological limits, which also "approximate" things, usually numbers or points in a space.

When we have a filtered or cofiltered diagram of subobjects $(A_\lambda)_{\lambda\in\Lambda}$ of a given object X and inclusion maps, the cofiltered limit gives the infimum ("intersection") of subspaces, and the filtered colimit gives the supremum ("union"). Whenever these subspaces are retracts ($\iota : A \to X, \pi : X \to A$), they give rise to idempotent morphisms $e = \iota \circ \pi : X \to X$, and if the category is Cauchy-complete, every idempotent arises in this way.

Using the canonical closed monoidal structure of Top, one can consider topologically enriched categories. In such an enriched category C, one can look if whenever a net of retracts $A_\lambda \subseteq X$ tends to a retract A as a *limit* (= infimum) or *colimit* (= supremum), the corresponding idempotents e_λ tend to e *topologically*, in the hom-space $\mathrm{C}(X, X)$. We call these properties, which may hold or fail depending on C, the *upward* and *downward Levi properties*, respectively for suprema and infima, in analogy with Beppo Levi's theorem, which says that every bounded monotone real sequence converges to its supremum.

An example of topologically enriched category where these properties hold is the category of Hilbert spaces and short maps [4, 5]. In this case, every (split) idempotent is the projection onto a closed subspace. Cofiltered limits of subspaces and inclusions give exactly intersections, and filtered colimits give closures of unions. A net of projectors (e_λ) onto subspaces A_λ tends topologically to a projector e if and only if the A_λ tend, as limit or colimit, to the subspace A splitting the idempotent e.

Another such category arises in probability theory, the category of standard Borel probability spaces and couplings between them [2]. Retracts, in this category, are sub-sigma-algebras up to almost sure equality. The upward and downward Levi properties hold, and give exactly convergence in mean for martingales and inverse martingales, cornerstone results of probability theory, which now have a categorical formalization and proof.

References

[1] Paolo Perrone and Ruben Van Belle, *Convergence of martingales via enriched dagger categories.* In preparation.

[2] Noé Ensarguet and Paolo Perrone, *Categorical probability spaces, ergodic decompositions, and transitions to equilibrium.* Submitted (2023), available at arXiv:2310.04267.

[3] Ruben Van Belle, *A categorical treatment of the Radon-Nikodym theorem and martingales.* Submitted (2023), available at arXiv:2305.03421.

[4] Chris Heunen and André Kornell, *Axioms for the category of Hilbert spaces.* Proceedings of the National Academy of Sciences USA 119(9), 2022.

[5] Matthew Di Meglio and Chris Heunen, *Dagger categories and the complex numbers: Axioms for the category of finite-dimensional Hilbert spaces and linear contractions.* Submitted (2024), available at arXiv:2401.06584.

A new centre for crossed modules

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Abstract.

Crossed modules are algebraic models of homotopy 2-types. By definition, a crossed module G_* is a group homomorphism $\partial : G_1 \to G_0$ together with an action of G_0 on G_1 satisfying some properties. The most important invariants of the crossed module G_* are the group $\pi_0(G_*) = Coker(\partial)$ and the $\pi_0(G_*)$-module $\pi_1(G_*) = Ker(\partial)$.

One of the main results of this paper is to show that any crossed module $\partial : G_1 \to G_0$ fits in a commutative diagram

$$\begin{tikzcd} G_1 \arrow[r, "\delta"] \arrow[d, "id"'] & \mathbf{Z}_0(G_*) \arrow[d, "\mathsf{z}_0"] \\ G_1 \arrow[r, "\partial"] & G_0 \end{tikzcd}$$

where the top horizontal $G_1 \xrightarrow{\delta} \mathbf{Z}_0(G_*)$ and right vertical $\mathbf{Z}_0(G_*) \xrightarrow{\mathsf{z}_0} G_0$ arrows have again crossed module structures. In fact, the first one is even a braided crossed module, which we call the *centre of the crossed module* $\partial : G_1 \to G_0$ and denote by $\mathbf{Z}_*(G_*)$.

We show that the braided monoidal category corresponding to the braided crossed module $\mathbf{Z}_*(G_*)$ is isomorphic to the Drinfeld centre of the monoidal category corresponding to G_*.

Our definition of $\mathbf{Z_0}(G_*)$ is based on certain crossed homomorphisms $G_0 \to G_1$ and has some advantage compared to one based on monoidal categories. Namely, the description of $\mathbf{Z}_*(G_*)$ in terms of crossed homomorphisms makes it easy to relate the centre of a crossed module to group cohomology. The essential invariants of $\mathbf{Z}_0(G_*)$ are closely related to low dimensional group cohomology. In fact, one has an isomorphism of groups $\pi_1(\mathbf{Z}_*(G_*)) \cong H^0(\pi_0(G_*), \pi_1(G_*))$ and the group $\pi_0(\mathbf{Z}_*(G_*))$ fits in an exact sequence

$$0 \to H^1(\pi_0(G_0), \pi_1(G_*)) \to \pi_0(\mathbf{Z}_*(G_*)) \to \mathsf{Z}_{\pi_1(G_*)}(\pi_0(G_*)) \to H^2(G_0, \pi_1(G_*)),$$

where $\mathsf{Z}_{\pi_1(G_*)}(\pi_0(G_*))$ is the subgroup of the centre of the group $\pi_0(G_*)$ consisting of those elements which act trivially on $\pi_1(G_*)$.

It should be pointed out that in the 80's Norrie also introduced the notion of a centre of a crossed module, but our notion differs from hers. Our centre can be shown to be a homotopy invariant, unlike hers.

I'll also briefly discuss a connection between this definition and the Gottlieb group of the classifying space of the crossed module.

The category of strong homotopy Lie Rinehart pairs

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Abstract.
The first example of a strong homotopy Lie algebroid was the BV-BRST complex. In modern language, it appears when homotopy transfer is applied to a resolution of a Lie Rinehart pair by a semi-free dgc algebra and a graded projective dg module. However, a homotopy theory in which this phenomena embeds does not yet exist: in the current homotopical algebra for Lie Rinehart pairs developed by J. Nuiten in [1], the base is fixed, hence the formalism does not address its semi-free resolution. Defining weak equivalences of strong homotopy Lie Rinehart pairs $(A, M) \to (B, N)$ as ∞-comorphisms in which maps $B \to A$ and $M \to A \otimes_B N$ are both quasi-isomorphisms, we show that the full subcategory of pairs (A, M) with A a semi-free dgc algebra and M a cofibrant A-module is a category of fibrant objects [2]. Apart from the above historical application, the formalism is expected to provide the correct notion of a cotangent complex of a general Lie Rinehart pair, enabling the study of shifted symplectic structures (introduced by Pym and Safranov for Lie algebroids over a smooth base [3]), derived Lagrangian intersections etc.

References

[1] J. Nuiten, *Homotopical algebra for Lie algebroids*, Appl. Categ. Struct., Vol. 27, 493–534, (2019)

[2] D. Pistalo, *The category of strong homotopy Lie Rinehart pairs*, In preparation

[3] B.Pym, P. Safronov *Shifted Symplectic Lie Algebroids*, Vol. 2020, 21 7489–7557, (2020)

The dichotomy between enriched and internal categorical structures

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Abstract.

We revisit the dichotomy between enriched and internal categories in a base category $\mathcal{V}$. By describing these objects as monads internal to certain proarrow equipments, we construct a change-of-base adjunction between the category of enriched $\mathcal{V}$-categories and the category of internal $\mathcal{V}$-categories, provided $\mathcal{V}$ satisfies suitable conditions. This perspective allows us to cast enriched $\mathcal{V}$-categories as internal $\mathcal{V}$-categories whose object-of-objects is discrete, which finds applications in the study of the descent theory for $\mathcal{V}$-functors, as shown in [3, Theorem 9.11].

Motivated by the study of the descent theory of functors between $(T, \mathcal{V})$-categories [1], the goal of [4] is to extend these techniques to the setting of generalized multicategories [2]. Using the above dichotomy as our guiding principle, we study the notion of change-of-base for a notion of lax algebras for monads in a suitable 2-category of proarrow equipments. Given suitable conditions on the category $\mathcal{V}$ and the monad T, we obtain an analogous adjunction between enriched and internal $(T, \mathcal{V})$-categories, which we use to describe effective descent functors of $(T, \mathcal{V})$-categories.

References

[CT03] M. M. Clementino and W. Tholen. Metric, topology and multicategory—a common approach. *J. Pure Appl. Algebra*, 179(1-2):13–47, 2003.

[CS10] G. Cruttwell and M. Shulman. A unified framework for generalized multicategories. *Theory Appl. Categ.*, 24(21):580–655, 2010.

[Luc18] F. Lucatelli Nunes. Pseudo-Kan extensions and descent theory. *Theory Appl. Categ.* **33**(15):390–444, 2018.

[PL23] R. Prezado and F. Lucatelli Nunes. Generalized multicategories: change-of-base, embedding and descent. Preprint 23-29, DMUC (2023).

The arrows between double category sites for Grothendieck topoi

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Abstract In [3] Ehresmann sites are introduced as a way to represent étendues by sites where the underlying categories are ordered groupoids. This result was proved through a correspondence between Ehresmann sites and left cancellative categories (which represent étendues, [2]). In [1], the notion of ordered groupoid was reinterpreted as a special type of double category: an internal groupoid in the category of posets, with the additional property that the source map from the groupoid of arrows to the groupoid of objects is a discrete fibration. The Ehresmann topology is defined on the vertical arrows but the horizontal arrows contribute to the conditions the covering families need to satisfy. This reinterpretation of Ehresmann sites gives rise to a *2-category* of Ehresmann sites such that the correspondence between left cancellative sites and Ehresmann sites given in [3] becomes part of:

- a 2-adjunction between the 2-category of left cancellative categories and the 2-category of ordered groupoids;
- a biequivalence between the 2-category of left cancellative categories and the 2-category of ordered groupoids where each connected component of the vertical category has a maximal object;
- a biequivalence between the 2-category of left cancellative Grothendieck sites and the 2-category of Ehresmann sites.

This last equivalence is obtained by characterizing the functors between Ehresmann sites that correspond to cover-preserving, covering-flat functors between Grothendieck sites and restricting the previous equivalence.

In our current work we introduce generalized Ehresmann sites that represent arbitrary Grothendieck topoi. Julia Ramos González will introduce the definition of generalized Ehresmann site in her talk and describe the type of Grothendieck sites we use to obtain an equivalence between the 2-categories of sites. The Grothendieck sites come with a left quadrable orthogonal factorization system where the

left class of arrows are covering in the Grothendieck topology and the right arrows are monic (we do not require the left class to contain all coverings and neither the right class to contain all monics).

I will present the generalization of the 2-adjunction just listed for left cancellative categories and Ehresmann sites to a 2-adjunction between the 2-category of categories with an orthogonal factorization system where the right class contains only monic arrows and the 2-category of ordered categories (double categories where the vertical structure is posetal and the domain functor is a discrete fibration). This 2-adjunction restricts to a biequivalence when we restrict ourselves to ordered categories where each component of the vertical category has a maximal object. (This result closely resembles the types of correspondences given in [4] but the way we define the double category from a factorization system is different: rather than taking the monic arrows themselves as vertical arrows, we take the subobjects they define.)

This gives us a good starting point to define the arrows between the generalized Ehresmann sites described by Julia in her talk: we have a correspondence on objects and we use cover-preserving, covering-flat arrows that preserve the factorization systems as arrows between the Grothendieck sites. The latter property assures us that we have corresponding double functors. So our job is now to translate the notions of being cover preserving and covering flat to maps between generalized Ehresmann sites. It is obvious what it means to be covering preserving, but the notion of covering flatness requires further work: which finite diagrams for an Ehresmann site should be used to test for covering-flatness? We need that cones over such finite diagrams for generalized Ehresmann sites correspond in a suitable fashion to cones for finite diagrams for our chosen Grothendieck sites.

To resolve this issue we show that each finite diagram $D\colon \mathcal{I} \to (\mathfrak{a}, J_{\mathfrak{a}})$ into a Grothendieck site with a suitable orthogonal factorization system factors through an indexing category $\widetilde{\mathcal{I}}$ with a strict factorization system so that the induced diagram is a functor of categories with factorization systems and such that cones for the first extend uniquely to cones for the second. The first correspondence in [4], for strict factorization systems, gives then the corresponding indexing double category, that gives us a finite diagram in the corresponding generalized Ehresmann site. This construction allows us to define covering flatness for maps between generalized Ehresmann sites in terms of finite diagrams indexed by finite double categories where the domain functor is a discrete fibration.

We now obtain a 2-category of generalized Ehresmann sites and covering-preserving covering-flat double functors (and Λ-transformations that form a straight generalization of those introduced in [1]). This 2-category is biequivalent to the 2-category of our chosen Grothendieck sites.

Finally, Comparison Lemma maps between our Grothendieck sites satisfy the conditions to form a bicategory of fractions. This allows us to obtain a class of Comparison Lemma maps between generalized Ehresmann sites so that Grothendieck topoi form the bicategory of fractions for generalized Ehresmann sites with respect to these morphisms.

References

[1] D. DeWolf, and D. Pronk, *A double categorical view on representations of étendues*, Cah. Topol. Géom. Différ. Catég. 61 (2020), no. 1, 3–56.

[2] A. Kock, and I. Moerdijk, *Presentations of étendues*, Cah. Topol. Géom. Différ. Catég. 32 (1991), no. 2, 145–164.

[3] M.V. Lawson, and B. Steinberg, *Ordered Groupoids and Étendues*, Cah. Topol. Géom. Différ. Catég. 45 (2004), no. 2, 82–108.

[4] Miloslav Štěpán, *Factorization systems and double categories*, arXiv:2305.06714v2 [math.CT]

Double categorical presentations of Grothendieck topoi

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Abstract. The orthogonal epi-mono factorization system in a Grothendieck topos together with the stability under pullbacks of epimorphisms (and monomorphisms) allows to fully encode the canonical topology on the topos, given by the jointly-epimorphic families, in terms of:

(M) the families of monic arrows that are jointly-epimorphic (i.e. the families of monic arrows that are covering);

(C) the epimorphic arrows (i.e. the single arrows that are covering);

(S) the stability under pullbacks of epimorphisms (and monomorphisms) along both epimorphisms and monomorphisms (i.e. single covering arrows and monic arrows satisfy the Ore condition).

On one hand, **(M)** points us towards the direction of left cancellative sites, i.e. sites where all morphisms are monic, and their categories of sheaves, the étendues. On the other hand, **(C)** + **(S)** point us towards the sites where all single arrows are covering, the atomic sites, and their categories of sheaves, the atomic topoi.

In this talk, inspired by this observation, we introduce the notion of *covering-mono Grothendieck site*, a small Grothendieck site endowed with an orthogonal factorization system in which the left class consists of single covering arrows and the right class consists of monomorphisms, together with suitable Ore conditions. We define *covering-mono morphisms of sites* to be the covering-flat covering-preserving morphisms that also preserve the factorization system. We then show that every Grothendieck topos admits a covering-mono site presentation and that the 2-category of Grothendieck topoi can be recovered as a bicategory of fractions of the 2-category of covering-mono sites, where one inverts the covering-mono morphisms that are LC (Lemme de Comparaison) [2]. Every Grothendieck topos can be seen in this way as an interpolation between an étendue and an atomic topos.

Étendues can also be presented in terms of Ehresmann sites [3], which are ordered groupoids endowed with an Ehresmann topology and that can be envisioned as suitable double categories where the Ehresmann topology lives in the vertical direction [1]. In the same way that covering-mono sites (presenting Grothendieck topoi) generalize left cancellative sites (presenting étendues), we introduce the

notion of *generalized Ehresmann site* (generalizing classical Ehresmann sites in their double categorical incarnation) and we show that every Grothendieck topos can be presented in terms of a generalized Ehresmann site. More concretely, we define a generalized Ehresmann site as a category internal in posets (which we envision as a double category), with suitable horizontal and 2-cellular Ore conditions and endowed vertically with an Ehresmann topology. The horizontal arrows are now not isomorphisms in general, as it was the case for the classical Ehresmann sites, but the Ore conditions imposed allow us to treat them as single covering arrows. In addition, we introduce the appropriate notion of sheaf on a generalized Ehresmann site, analogous to the notion of sheaf on a classical Ehresmann site. We then establish a direct connection between covering-mono Grothendieck sites and generalized Ehresmann sites enhancing the connection between left cancellative sites and classical Ehresmann sites from [3] and [1]: to each covering-mono Grothendieck site we associate a generalized Ehresmann site and vice versa, and we show that these associations respect the operation of taking sheaves. This allows us to conclude that generalized Ehresmann sites provide presentations for Grothendieck topoi, as desired.

We will present the relation between covering-mono Grothendieck sites and generalized Ehresmann sites exclusively at the level of objects. However, this relation can be understood at the level of bicategories (as it is the case for left cancellative sites and classical Ehresmann sites, see [3] and [1]). The relation at the bicategorical level and the recovery of the 2-category of Grothendieck topoi as a bicategory of fractions of the 2-category of generalized Ehresmann sites by inverting the corresponding class of LC morphisms of generalized Ehresmann sites will not be treated in this talk, but will be presented by Dorette Pronk in hers.

While LC morphisms of sites (resp. LC covering-mono morphisms of sites) admit a calculus of fractions [4] allowing to recover the 2-category of Grothendieck topoi as a bicategory of fractions of the 2-category of sites (resp. of the 2-category of covering-mono sites), the 2-categories of left cancellative sites and classical Ehresmann sites are too restrictive in order for the LC morphisms to admit a calculus of fractions. In the last part of the talk, we identify larger families of presentations of étendues that solve this issue. More concretely, we identify a subclass of the covering-mono sites, the *torsion-free generated covering-mono sites*, enlarging the subclass of left cancellative sites but still providing presentations of étendues. In parallel, we describe the corresponding subclass of the generalized Ehresmann sites, which we call the *torsion-free generated generalized Ehresmann sites*. We show that the LC morphisms in the class of torsion-free generated covering-mono sites do admit a calculus of fractions allowing to recover the 2-category of étendues as a bicategory of fractions. Through the bicategorical relation between covering-mono sites and generalized Ehresmann sites that Dorette Pronk will present in her talk, it is immediate to obtain the analogous result for the torsion-free generated generalized Ehresmann sites.

References

[1] D. DeWolf, and D. Pronk, *A double categorical view on representations of étendues*, Cah. Topol. Géom. Différ. Catég. 61 (2020), no. 1, 3–56.

[2] A. Kock, and I. Moerdijk, *Presentations of étendues*, Cah. Topol. Géom. Différ. Catég. 32 (1991), no. 2, 145–164.

[3] M.V. Lawson, and B. Steinberg, *Ordered Groupoids and Étendues*, Cah. Topol. Géom. Différ. Catég. 45 (2004), no. 2, 82–108.

[4] J. Ramos González, *Grothendieck categories as a bilocalization of linear sites*, Appl. Categ. Structures 26 (2018), no. 4, 717–745.

Modules over invertible 1-cocycles

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Abstract.
Hopf braces are recent mathematical objects introduced by I. Angiono, C. Galindo and L. Vendramin in [1] and obtained through a linearisation process from skew braces, which give rise to non-degenerate, bijective and not necessarily involutive solutions of the Quantum Yang-Baxter Equation (see [5]), whose formulation is the following:

$$(\tau \otimes id_V) \circ (id_V \otimes \tau) \circ (\tau \otimes id_V) = (id_V \otimes \tau) \circ (\tau \otimes id_V) \circ (id_V \otimes \tau), \tag{QYBE}$$

where $\tau \colon V \otimes V \to V \otimes V$ is a linear map and V, a $\mathbb{K}$-vector space. As was proven in [1, Corollary 2.4], cocommutative Hopf braces are also relevant from a physical standpoint because they also induce solutions of the above-mentioned equation.

On the one hand, a well-known result for Hopf braces is their strong relationship with invertible 1-cocycles due to the fact that both categories are equivalent (see [1, Theorem 1.12] and [4, Theorem 3.2]). These objects are no more than coalgebra isomorphism between Hopf algebras $\pi \colon H \to B$, related to each other through a module-algebra structure, and satisfying a weaker condition than being algebra morphism.

On the other hand, R. González Rodríguez in [3] introduced the notions of module over a Hopf brace and Hopf module over a Hopf brace, obtaining a categorical equivalence between the base braided monoidal category C and the category of Hopf modules over a Hopf brace, also known by the Fundamental Theorem of Hopf modules for Hopf braces.

Therefore, considering the aforementioned precedents, the aim of this talk is going to be giving a suitable notion of module over a invertible 1-cocycle in such a way that the categorical equivalence between Hopf braces and invertible 1-cocycles remains valid between their module categories.

References

[1] I. Angiono, C. Galindo and L. Vendramin, Hopf braces and Yang-Baxter operators, *Proc. Am. Math. Soc.* **145**(5) (2017) 1981-1995.

[2] J. M. Fernández Vilaboa, R. González Rodríguez, B. Ramos Pérez and A. B. Rodríguez Raposo, Modules for invertible 1-cocycles, Preprint (2023), arXiv:2311.05233 **[math.RA]**.

[3] R. González Rodríguez, The fundamental theorem of Hopf modules for Hopf braces, *Linear Multilinear Algebra* **70**(20) (2022) 5146-5156.

[4] R. González Rodríguez and A. B. Rodríguez Raposo, Categorical equivalences for Hopf trusses and their modules, Preprint (2023), arXiv:2312.06520 **[math.RA]**.

[5] L. Guarnieri and L. Vendramin, Skew braces and the Yang–Baxter equation, *Math. Comput.* **86**(307) (2017) 2519-2534.

Lawvere Theories and Symmetric Operads as Substitution Algebras: Free constructions for Abstract Syntax

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Abstract. We are interested in free constructions of Lawvere theories and of symmetric operads from syntactic structures. These respectively correspond to the syntax of cartesian and symmetric monoidal second-order theories, and most generally arise from binding signatures. For a direct constructive approach to these free constructions, it is best to consider Lawvere theories and symmetric operads as *substitution algebras* modelling single-variable substitution.

In the cartesian case, one considers the object-classifier topos $\mathcal{F} = \mathbf{Set}^{\mathbb{F}}$ (where $\mathbb{F}$ is the category of finite cardinals and functions) as the ambient category for models. Here, a substitution algebra [4] consists of a presheaf, $X \in \mathcal{F}$, a variable operation $\nu : 1 \to \delta(X)$, and a single-variable substitution operation $\sigma : \delta(X) \times X \to X$ satisfying the following four axioms:

$$\begin{tikzcd} 1 \times X \arrow[r, "\pi_2"] \arrow[d, "\nu \times \mathrm{id}"'] & X \\ \Sigma_{\mathrm{s}}(X) \arrow[ur, "\sigma"'] & \end{tikzcd} \qquad \begin{tikzcd} X \times X \arrow[r, "\pi_1"] \arrow[d, "\mathrm{up}_X \times \mathrm{id}"'] & X \\ \Sigma_{\mathrm{s}}(X) \arrow[ur, "\sigma"'] & \end{tikzcd} \qquad \begin{tikzcd} \delta^2(X) \times 1 \arrow[rr, "\pi_1"] \arrow[d, "\mathrm{id} \times \nu"'] & & \delta^2(X) \arrow[d, "\mathrm{cont}"] \\ \delta^2(X) \times \delta(X) \arrow[r, "\cong"] & \delta\Sigma_{\mathrm{s}}(X) \arrow[r, "\delta(\sigma)"] & \delta(X) \end{tikzcd}$$

$$\begin{tikzcd} \delta\Sigma_{\mathrm{s}}(X) \times X \arrow[r, "\cong"] \arrow[d, "\delta(\sigma) \times \mathrm{id}"'] & \Sigma_{\mathrm{s}}\delta(X) \times X \arrow[r, "\mathbf{str}"] & \Sigma_{\mathrm{s}}\Sigma_{\mathrm{s}}(X) \arrow[r, "\Sigma_{\mathrm{s}}(\sigma)"] & \Sigma_{\mathrm{s}}(X) \arrow[d, "\sigma"] \\ \Sigma_{\mathrm{s}}(X) \arrow[rrr, "\sigma"] & & & X \end{tikzcd}$$

In the above definition, $(\delta, \mathrm{up}, \mathrm{cont}, \mathrm{swap})$ is a *symmetric monad* on $\mathcal{F}$, induced by the structure of $\mathbb{F}$, and $\Sigma_{\mathrm{s}}(A) = \delta(A) \times A$ is a strong endofunctor on $\mathcal{F}$. Substitution algebras are equivalent to Lawvere theories [4] and provide a finite equational presentation of Lawvere theories over $\mathcal{F}$ [5], in contrast to their countably-sorted presentation as abstract clones.

Furthermore, $\mathcal{F}$ is the suitable environment for second-order cartesian theories, conservatively extending Lawvere theories [1]. Binding signatures account for algebraic operations with variable binding; they appear, for example, as abstraction in the lambda calculus and quantifiers in predicate logic [3]. An endofunctor Σ on $\mathcal{F}$ constructed using the product, coproduct, and δ may be associated with each binding signature. We prove that the free Σ-algebra over the presheaf of abstract variables, $\mathcal{Y}(1) = \mathbb{F}(1, -) : \mathbb{F} \hookrightarrow \mathbf{Set}$, is equipped with a canonical substitution algebra structure which is

induced by *generalised parametrised structural recursion.* This structure models the abstract syntax of the binding signature and is initial in the category of Σ-algebras with compatible substitution algebra structure.

For symmetric monoidal theories – for which the first-order theories are symmetric operads – the appropriate ambient category for models is that of species of structures, $\mathcal{B} = \mathbf{Set}^{\mathbb{B}}$, where $\mathbb{B}$ is the groupoid of finite cardinals [6, 7]. $\mathcal{B}$ has an additional monoidal tensor, namely the Day tensor product, $\otimes$. This is used, instead of the cartesian product, to model linear pairing. The analogous δ on $\mathcal{B}$ is only a *symmetric endofunctor* and does not respect linear pairings as in the cartesian case. Instead, it is a derivative operator, equipped with a *Leibniz* canonical natural isomorphism, $\delta(A) \otimes B + A \otimes \delta(B) \xrightarrow{\cong} \delta(A \otimes B)$.

An endofunctor on $\mathcal{B}$ for a binding signature Σ is constructed using the Day tensor, coproduct, and the derivative δ. Using the Leibniz isomorphism, we define a *derived* functor $\Sigma' : \mathcal{B}^2 \to \mathcal{B}$ together with a canonical isomorphism $\delta\Sigma(A) \cong \Sigma'(A, \delta(A))$.

We define a linear substitution algebra (equivalent to that of [2]) as a presheaf $Y \in \mathcal{B}$ together with a variable operation $v : I \to \delta(Y)$ and a single-variable substitution operation $\varsigma : \delta(Y) \otimes Y \to Y$ satisfying the following two axioms:

$$
\begin{tikzcd}
I \otimes Y \arrow[r, "\cong"] \arrow[d, "v\otimes\mathrm{id}"'] & Y \\
\delta(Y) \otimes Y \arrow[ur, "\varsigma"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\delta\Sigma_{\mathrm{s}}(Y) \otimes Y \arrow[r, "\cong"] \arrow[d, "\delta(\varsigma)\otimes\mathrm{id}"'] & \Sigma'_{\mathrm{s}}(Y, \delta(Y)) \otimes Y \arrow[r, "\mathbf{str}_{\mathrm{s}}"] & \Sigma'_{\mathrm{s}}(Y, \delta(Y) \otimes Y) \arrow[r, "\Sigma'_{\mathrm{s}}(\mathrm{id},\varsigma)"] & \Sigma'_{\mathrm{s}}(Y, Y) \arrow[d, "\varsigma+\varsigma"] \\
\delta(Y) \otimes Y \arrow[rrr, "\varsigma"] & & & Y
\end{tikzcd}
$$

Here, $\Sigma_{\mathrm{s}}(A) = \delta(A) \otimes A$ and $\Sigma'_{\mathrm{s}}(A, B) = \delta(B) \otimes A + \delta(A) \otimes B$. The category of linear substitution algebras is equivalent to the category of symmetric operads (and, indeed, the category of simultaneous-substitution monoids [8]).

We prove that the free Σ-algebra over $\mathcal{Y}(1) = \mathbb{B}(1, -) : \mathbb{B} \to \mathbf{Set}$ has an induced linear substitution algebra that is initial in the category of Σ-algebras with compatible linear substitution algebra structure. The full study of second-order symmetric monoidal theories remains work in progress.

References

[1] M. Fiore, *Second-order and dependently-sorted abstract syntax*, Proceedings of the 23rd Symposium on Logic in Computer Science, 57–68, 2008.

[2] M. Fiore, *Lie structure and composition*, CT2014 slides, 2014.

[3] M. Fiore, O. Mahmoud, *Second-Order Algebraic Theories*, Preprint, arXiv:1308.5409, 2013.

[4] M. Fiore, G. Plotkin, D. Turi, *Abstract syntax and variable binding*, Proceedings of the 14th Symposium on Logic in Computer Science, 193–202, 1999.

[5] M. Fiore, S. Ranchod *A finite algebraic presentation of Lawvere theories in the object-classifier topos*, Submitted, 2024.

[6] A. Joyal, *Une théorie combinatoire des séries formelles*, Advances in Mathematics, 42:1–82, 1981.

[7] A. Joyal, *Foncteurs analytiques et espèces de structures*, Combinatoire énumérative, Lecture Notes in Mathematics, vol. 1234, 126–159, 1986.

[8] G.M. Kelly, *On the operads of J.P. May*, Reprints in Theory and Applications of Categories, No. 13, pp. 1–13, 2005.

Categories of modules, comodules and contramodules over representations

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Abstract.

In [2], Estrada and Virili considered a representation (a functor) $\mathcal{A} : \mathscr{C} \longrightarrow Add$ of a small category $\mathscr{C}$ taking values in Add of small preadditive categories and introduced a concept of modules over such a representation. A sheaf of $\mathcal{O}_X$-modules over a scheme $(X, \mathcal{O}_X)$ is the prototypical example of a module over such a representation.

In algebraic geometry, the idea of studying schemes by means of module categories linked with adjoint pairs given by extension and restriction of scalars is well developed in the literature. In this talk, we will consider representations of a small category taking values in (co)algebras and build an algebraic geometry like categorical framework that studies modules, comodules and contramodules over such representations using adjoint functors. We will discuss the cartesian objects in each of these categories, which may be viewed as counterparts of quasi-coherent sheaves over a scheme. We will focus on understanding the generators for these categories and the Grothendieck categories appearing in these contexts, because the latter may be treated as replacements for noncommutative spaces. This is a recent joint work [1] with Balodi and Banerjee.

References

[1] M. Balodi, A. Banerjee, S. Ray, *Categories of modules, comodules and contramodules over representations*, Forum Mathematicum, Vol.**36**, No.1, 111–152 (2023).

[2] S. Estrada, S. Virili, *Cartesian modules over representations of small categories*, Advances in Mathematics, Vol.**310**, 557–609 (2017).

Cotraces and Inner Product Enrichment of Bicategories

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Abstract. The worlds of category theory and linear algebra seem to be closely, but informally, interwoven. On the one hand, many category theoretic contructions have strong analogues in linear algebra. Most obvious, perhaps, is the similarity between categorical and linear adjoints. But it is often useful to find other comparisons as well: categorical limits might be compared to Cauchy limits, or products; profunctors might be compared to matrices; the Yoneda lemma might be compared to the Riesz representation theorem.

On the other hand, category theory has been proven to be a useful tool in the abstraction of linear algebraic results. Compact closed categories, dagger categories, scalars and traces in monoidal categories all provide valuable abstractions that allow us to do away with matrix computation and work much more intuitively with string diagrams. For the particular case of dagger compact categories see, for example, work of Doplicher and Roberts [1], Baez and Dolan [2], Abramsky and Coecke [3], Selinger [4], and Heunen and Kornell [5].

We are interested here in the varied and subtle roles that scalars, traces and inner products play in the world of *bi*categories. In particular, trace-like structures are well-studied. Thinking of bicategories as generalised monoidal categories, Ponto [6] gave the structure and conditions necessary to take the trace of an endo-2-cell, and this formalism has been used for a number of applications in topology. Ponto and Shulman [7] later studied the various category theoretic properties of this notion of trace. Much more recently, Hess and Rasekh [8] related this form of trace to topological Hochschild homology. Work by Bartlett [9], and Ganter and Kapranov [10], gave the definition of the 'categorical trace' or '2-trace' which is defined by considering the bicategory of 2-Vector spaces. And as far back as 1997, Day and Street [11] published an account of compact closed bicategories in which they gave the definition of a *co*trace.

Our new work shows that this cotrace has an incredibly rich structure and enjoys a number of trace-like properties. What's more, it can be used to define a sort of categorical inner product – defined analogously to the Frobenius inner product – which gives an enrichment to the whole bicategory. In the same way that the Frobenius inner product is a scalar that exists 'between' two linear maps, this scalar enrichment replaces every set of 2-cells with a categorical scalar.

This result has several consequences. Firstly, it highlights a canonical enrichment that gives many well-known bicategories their extra structure. For example, it allows us to replace sets of enriched natural transformations with their corresponding natural transformation object.

Secondly, it unifies Day and Street's cotrace with the 'categorical trace' as defined by Ganter and Kapranov and the '2-trace' as defined by Bartlett for 2-Hilbert spaces. It turns out that the 2-trace is simply the unenriched cotrace.

Thirdly, it provides a theoretical underpinning for Willerton's [12] observation that the 2-trace seems to be somehow dual to the usual notion of trace. Willerton pointed out that, if we extend the

definitions of trace and 2-trace to the context of a bicategory with duals, these two different traces often appear to give opposing results. For example, in the bicategory of profunctors the trace gives a coend, but the 2-trace gives an end; in a particular bicategory of bimodules, the trace gives Hochschild homology, but the 2-trace gives Hochschild cohomology. The problem with this observation was that the trace is a scalar – that is, a map from the unit object to itself – whereas the 2-trace is a *set* of 2-cells. It is only after adding appropriate enrichment that this ostensible duality makes sense.

Finally, it is a further step towards formalising the relationship between categorical adjoints and adjoints of linear maps. Since the enrichment is defined, and behaves, like an inner product, 1-cells that are adjoint in the category theoretic sense are also adjoint in a linear sense.

In this talk we will explore how the enrichment is defined, the similarities that exist between the enrichment and the Frobenius inner product, the trace-like properties that this endows the cotrace with, and the many bicategories for which the trace and cotrace give interesting and dual constructions.

References

[1] S. Doplicher, J. E. Roberts, *A new duality theory for compact groups*, Inventiones Mathematicae 98 (1989), no. 1.

[2] J. C. Baez, J. Dolan, *Higher-dimensional algebra and topological quantum field theory*, Journal of Mathematical Physics 36 (1995), no. 11.

[3] S. Abramsky, B. Coecke, *A categorical semantics of quantum protocols*, Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science (2004).

[4] P. Selinger, *Finite dimensional Hilbert spaces are complete for dagger compact closed categories*, Logical Methods in Computer Science 8 (2012), no. 3.

[5] C. Heunen, A. Kornell, *Axioms for the category of Hilbert spaces*, Proceedings of the National Academy of Sciences of the United States of America 119 (2022), no. 9.

[6] K. Ponto, *Fixed point theory and trace for bicategories*, Astérisque (2010), no. 333.

[7] K. Ponto, M. Shulman, *Shadows and traces in bicategories*, Journal of Homotopy and Related Structures 8 (2013).

[8] K. Hess, N. Rasekhm, Shadows are bicategorical traces, preprint arXiv:2109.02144, 2021.

[9] B. Bartlett, On unitary 2-representations of finite groups and topological quantum field theory, PhD thesis, University of Sheffield, 2009

[10] N. Ganter, M. Kapranov, *Representation and character theory in 2-categories*, Advances in Mathematics 217 (2007), no. 5.

[11] B. Day, R. Street, *Monoidal bicategories and Hopf algebroids*, Advances in Mathematics 129 (1997), no. 1.

[12] S. Willerton, Two 2-traces, Conference talk: Logic and Foundations of Physics VII, University of Birmingham, 2010.

Adjoint split extensions of categories

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Abstract.

Suppose we are given a parameterized monad, meaning a functor $\mathcal{B} \to \mathsf{Mnd}(\mathcal{X})$ from $\mathcal{B}$ to the category of monads on $\mathcal{X}$. This can be interpreted as a particular kind of action of the category $\mathcal{B}$ on the category $\mathcal{X}$. The data of the functor $\mathcal{B} \to \mathsf{Mnd}(\mathcal{X})$ can also be given as a functor

$$\mathcal{B} \times \mathcal{X} \to \mathcal{X}\,, \quad (B, X) \mapsto B \cdot X\,,$$

where each endofunctor $B \cdot -$ carries the structure of a monad.

To each monad $B \cdot -$ we can associate its category of algebras, and when we glue these categories together using the Grothendieck construction, we get the category $\mathcal{X} \rtimes \mathcal{B}$, whose objects are triples

$$(X, B, B \cdot X \xrightarrow{\xi} X)\,,$$

with $X \in \mathsf{Ob}(\mathcal{X})$, $B \in \mathsf{Ob}(\mathcal{B})$ and ξ a $(B \cdot -)$-algebra on X. We treat an algebra $B \cdot X \xrightarrow{\xi} X$ as the data of an action the object B on the object X, so $\mathcal{X} \rtimes \mathcal{B}$ can be considered as a category of actions, with the parameterized monad $\mathcal{B} \to \mathsf{Mnd}(\mathcal{X})$ specifying what it means for $B \in \mathsf{Ob}(\mathcal{B})$ to act on $X \in \mathsf{Ob}(\mathcal{X})$.

The category $\mathcal{X} \rtimes \mathcal{B}$ has an associated fibration

$$p\colon \mathcal{X} \rtimes \mathcal{B} \to \mathcal{B}\,, \quad (X, B, \xi) \mapsto B$$

and, as long as $\mathcal{X}$ and $\mathcal{B}$ have initial and terminal objects and the monad $\mathbf{0} \cdot -$ corresponding to the initial object $\mathbf{0} \in \mathsf{Ob}(\mathcal{B})$ is the identity monad, we can form the diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r, hook, shift left=1.5, "i"] \arrow[r, phantom, "\perp"] & \mathcal{X} \rtimes \mathcal{B} \arrow[l, shift left=1.5, "i^R"] \arrow[r, "p" description] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathcal{B} \arrow[l, hook', shift right=3, "s"'] \arrow[l, hook', shift left=3, "p^R"]
\end{tikzcd}$$

of categories and adjunctions. We will argue that this diagram is an **adjoint split extension** in a suitable setting, with the word *adjoint* indicating that the splitting s of p is left adjoint to p.

In fact, we will treat this diagram as the archetypal adjoint split extension, and describe a theory of such extensions, where the $\mathcal{X} \rtimes \mathcal{B}$ construction will play the role of the semi-direct product that we construct from the action $\mathcal{B} \to \mathsf{Mnd}(\mathcal{X})$. We view these extension in the setting where the morphisms between categories are adjunctions, meaning we need to specify what *exactness* means for a sequence

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook, shift left, "i"] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, shift left, "i^R"] \arrow[r, shift left, "p"] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, shift left, "p^R"] \end{tikzcd}$$

whose composite is the zero adjunction, meaning the adjunction between the constantly initial and constantly terminal functors. This exactness turns out to be approximately equivalent to saying that the full subcategories $\mathcal{X}$ and $\mathcal{B}$ form a torsion theory in $\mathcal{A}$.

For any adjoint split extension

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook, shift left, "i"] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, shift left, "i^R"] \arrow[r, "p" description] & \mathcal{B} \arrow[l, bend right, "s"'] \arrow[l, bend left, "p^R"] \end{tikzcd}$$

we have a comparison $\mathcal{A} \to \mathcal{X} \rtimes \mathcal{B}$, which, under suitable extra assumptions on the diagram, is an equivalence. For example, we get $\mathcal{A} \to \mathcal{X} \rtimes \mathcal{B}$ when our categories are the opposites of toposes, as observed in [3], in which case $\mathcal{X} \rtimes \mathcal{B}$ is just Artin gluing. We can also show that $\mathcal{A} \simeq \mathcal{X} \rtimes \mathcal{B}$ when we are working with semi-abelian categories, and, for example, [4] can be seen as making use of the equivalence $\mathsf{ccHopf}_{\mathbb{K}} \simeq \mathsf{Lie}_{\mathbb{K}} \rtimes \mathsf{Group}$, where $\mathsf{ccHopf}_{\mathbb{K}}$ is the category of cocommutative Hopf algebras over a nice field $\mathbb{K}$.

In categorical algebra, one often encounters the functor

$$\flat \colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}\,, \quad (B, X) \mapsto B \flat X := \ker(X + B \xrightarrow{[0,1]} B)$$

for a pointed category $\mathcal{A}$ with coproducts and pullbacks, and from our point of view this functor is the action associated to the adjoint split extension of points

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook, shift left, "i"] \arrow[r, phantom, "\perp"] & \mathsf{Pt}_{\mathcal{A}} \arrow[l, shift left, "i^R"] \arrow[r, "p" description] & \mathcal{A}\,. \arrow[l, bend right, "s=p^R"'] \arrow[l, bend left, "p^R"] \end{tikzcd}$$

The properties of the comparison $\mathsf{Pt}_{\mathcal{A}} \to \mathcal{A} \rtimes \mathcal{A}$ can be used to characterize the properties of $\mathcal{A}$. For example, conservativity of the comparison can be used to characterize the *protomodularity* of $\mathcal{A}$, and being an equivalence means that $\mathcal{A}$ is a *category with semidirect products* in the sense of [2].

References

[1] F. Borceux, G. Janelidze, G. Kelly, *Internal object actions*, Comment. Math. Univ. Carolin. 46 (2005), no. 2, 235–255.

[2] D. Bourn, G. Janelidze, *Protomodularity, descent, and semidirect products*, Theory Appl. Categ. 4 (1998), No. 2, 37–46.

[3] P. Faul, G. Manuell, and J. Siqueira, *Artin glueings of toposes as adjoint split extensions*, J. Pure Appl. Algebra 227 (2023), no. 5.

[4] M. Gran, G. Kadjo, J. Vercruysse, *A torsion theory in the category of cocommutative Hopf algebras*, Appl. Categ. Structures 24 (2016), no. 3, 269-282.

Bigraded path homology and the magnitude-path spectral sequence

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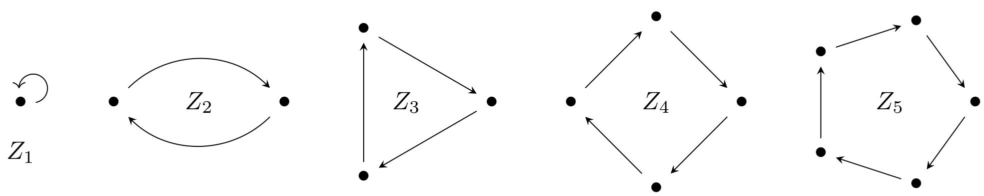
Abstract. The past two decades have seen a proliferation of homology theories for graphs (directed and undirected), including discrete and cubical homology, path homology, magnitude homology and reachability homology. Each of these theories is homotopy-invariant in an appropriate sense and satisfies some sensible discrete analogue of the Eilenberg–Steenrod axioms. Despite this, they tend to disagree even on quite primitive classes of graphs. For instance, magnitude homology distinguishes the directed cycles Z_m for every $m \geq 1$; path homology sees Z_1 and Z_2 as 'contractible' and all the rest as 'circle-like'; and to reachability homology every directed cycle appears contractible.



Thus, the evolving story of the homology of graphs is not a simple retelling of the classical story for spaces (in which Eilenberg–Steenrod's axiomatization guarantees a theory that is essentially unique). This talk, based on [3] and [4], presents a new chapter in the tale.

The diversity of homological viewpoints has motivated a recent drive towards consolidation using the framework of formal homotopy theory. Results so far have mainly been negative: for various natural notions of weak equivalence of graphs, it is known that no model structure can exist. But there has been one positive result: Carranza *et al* [2] exhibit a cofibration category structure on the category of directed graphs, for which the weak equivalences are maps inducing isomorphisms on path homology. One of the interesting features of their work is the choice of cofibrations, whose definition is reminiscent of the pairs of spaces for which the Mayer–Vietoris theorem holds in magnitude homology. As it turns out, this is no coincidence. Asao has shown that the two homology theories are closely related, appearing on consecutive pages in a certain spectral sequence [1].

The talk will describe ongoing work to understand that sequence, now known as the *magnitude-path spectral sequence* (or MPSS) of a directed graph. We refer to the underlying filtered chain complex as the *reachability complex*; its homology—the target object of the MPSS—is reachability homology.

Page 1 of the MPSS is exactly magnitude homology, while page 2 is a natural refinement of path homology, which lies its horizontal axis; we call this page the *bigraded path homology* of a directed graph. The sequence thus encompasses several existing invariants, and clarifies the relationships between them, while adding infinitely many new ones. The invariance properties of the pages grow progressively stronger as one passes through the sequence, giving rise to a nested family of weak equivalence classes of directed graphs. For instance, page r of the MPSS sees the directed m-cycle Z_m as contractible when m is less than r, and distinguishes each of the Z_ms for $m \geq r$.

Concerning the spectral sequence as a whole, our main results are as follows.

Theorems *Every page of the MPSS has the following homological properties:*

- *It satisfies an excision theorem with respect to the cofibrations in* [2].
- *It satisfies a Künneth theorem with respect to the box product.*
- *It is a finitary functor on the category of directed graphs (meaning it preserves filtered colimits).*

In particular these hold for bigraded path homology, which also satisfies a Mayer–Vietoris theorem.

This allows us to show that the cofibration category structure in [2] admits a natural refinement.

Theorem *The category of directed graphs carries a cofibration category structure in which the cofibrations are those of* [2] *and the weak equivalences are maps inducing isomorphisms on bigraded path homology. This is a strictly finer structure than the one exhibited in* [2]*: for instance, bigraded path homology, unlike ordinary path homology, distinguishes the directed m-cycles for every $m \geq 2$.*

These results have consequences of three sorts. Firstly, they demonstrate the value of bigraded path homology as a novel invariant of directed graphs, sharing the good properties of ordinary path homology, but with greater distinguishing power. Thus, in applications where path homology might be used, it is worth considering the bigraded variant.

Secondly, on a technical front, our methods illustrate a useful principle: that properties of path homology are frequently (though not always) 'inherited' from corresponding properties of magnitude homology—and that what holds true for either of these will often hold true throughout the MPSS. Moreover, arguments at the level of the reachability complex tend to be more straightforward than the rather involved proofs necessitated by the standard construction of path homology. Thus, when trying to prove statements about path homology, it is worth considering whether they can be approached via the spectral sequence.

Finally, we hope the MPSS will eventually cast more light on the homotopy theory of directed graphs. It is tempting to speculate that the cofibration category we describe may belong to a nested family of structures, one for each page; time permitting, we will sketch this idea at the end of the talk.

References

[1] Y. Asao, *Magnitude homology and path homology*, Bull. Lond. Math. Soc. 55 (2023), no. 1, 375–398.

[2] D. Carranza, B. Docherty, K. Kapulkin, M. Opie, M. Sarazola, and L. Ze Wong, *Cofibration category of digraphs for path homology*, Algebr. Comb., to appear.

[3] R. Hepworth and E. Roff, The reachability homology of a directed graph, preprint arXiv:2312.01378, 2023.

[4] R. Hepworth and E. Roff, Bigraded path homology and the magnitude-path spectral sequence, in preparation.

Homotopy Bicategories of Complete 2-fold Segal Spaces

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Abstract.

Across the multitude of definitions for a higher category, a dividing line can be found between two major camps of model. On one side lives the 'algebraic' models, like Bénabou's bicategories, tricategories following Gurski and the models of Batanin and Leinster [6, ch. 9], Trimble [3] and Penon [9]. These models specify composition operations and higher coherence morphisms, such as associators, all satisfying certain coherence conditions, such as the pentagon condition for bicategories. On the other end, one finds the 'non-algebraic' models, which do not make such specifications and may instead allow many choices of composite. These include the models of Tamsamani [11] and Paoli [8], along with quasicategories [4], complete n-fold Segal spaces [1] and more.

The bridges between these models remain somewhat mysterious. Progress has been made in certain instances, as seen in the work of Tamsamani [11], Lack and Paoli [5], Campbell [2], Moser [7] and others. Nonetheless, the correspondence remains incomplete; indeed, for instance, there is no fully verified means in the literature to take a weak 'algebraic' homotopy n-category of any known model of weak (∞, n)-category for general n.

In this talk, I will present a concrete means to explicitly construct homotopy bicategories of non-algebraic $(\infty, 2)$-categories, in particular Reedy fibrant complete 2-fold Segal spaces. The method, developed in [10], relies on choosing solutions to certain lifting problems, which determine choices of horizontal composition operations. Homotopies between such solutions induce associators and unitors, while coherence conditions are finally induced by higher homotopies between these homotopies. The methods presented extend neatly to obtaining pseudofunctors from maps between complete 2-fold Segal spaces as well. I will compare this construction to other ways one may obtain homotopy bicategories of complete 2-fold Segal spaces elsewhere in the literature, including methods induced by the work of Campbell [2] and of Moser [7].

References

[1] Clark Barwick. *(∞, n)-Cat as a Closed Model Category.* PhD thesis, University of Pennsylvania, 2005.

[2] Alexander Campbell. A homotopy coherent cellular nerve for bicategories. *Advances in Mathematics*, 368:107138, July 2020.

[3] Eugenia Cheng. Comparing operadic theories of n-category. *Homology, Homotopy and Applications*, 13:217–249, 2008.

[4] A. Joyal. Quasi-categories and Kan complexes. *Journal of Pure and Applied Algebra*, 175(1):207–222, November 2002.

[5] Stephen Lack and Simona Paoli. 2-nerves for bicategories. *K-Theory*, 38(2):153–175, January 2008.

[6] Tom Leinster. *Higher operads, higher categories*. Number 298 in London Mathematical Society lecture note series. Cambridge University Press, Cambridge, UK ; New York, 2004.

[7] Lyne Moser. A double $(\infty,1)$-categorical nerve for double categories, May 2021.

[8] Simona Paoli. Weakly globular N-fold categories as a model of weak N-categories, September 2016.

[9] Jacques Penon. Approche polygraphique des ∞-catégories non strictes. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 40(1):31–80, 1999.

[10] Jack Romö. Homotopy bicategories of complete 2-fold segal spaces, 2023.

[11] Zouhair Tamsamani. Sur des notions de n—categorie et n—groupoide non strictes via des ensembles multi-simpliciaux (On the notions of a nonstrict n–category and n–groupoid via multi-simplicial sets). *K-Theory*, 16(1):51–99, 1999.

Enriched Grothendieck topologies under change of base.

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Abstract.
In the presence of a monoidal adjunction

$$F \dashv G : \mathcal{U} \leftrightarrows \mathcal{V}$$

between locally finitely presentable Bénabou cosmoi, we examine the behavior of $\mathcal{V}$-Grothendieck topologies on a $\mathcal{V}$-category $\mathcal{C}$, and that of their constituent covering sieves, under the change of enriching category

$$G_* : \mathcal{V}\text{-Cat} \to \mathcal{U}\text{-Cat}$$

induced by G. In particular, we prove some basic lattice-theoretic properties of the collection of $\mathcal{V}$-Grothendieck topologies on $\mathcal{C}$, and that when G is faithful and conservative, any $\mathcal{V}$-Grothendieck topology on $\mathcal{C}$ corresponds uniquely to a $\mathcal{U}$-Grothendieck topology on $G_*\mathcal{C}$.

References

[1] Borceux, F., Quinteiro, C. (1996). A theory of enriched sheaves. Cahiers de Topologie et Géométrie Différentielle Catégoriques, 37(2), 145–162.

[2] Reyes, M. L. (2024). The finite dual coalgebra as a quantization of the maximal spectrum. Journal of Algebra, 644, 287–328.

[3] van den Berg, B., Heunen, C. (2014). Extending obstructions to noncommutative functorial spectra. Theory and Applications of Categories, 29, No. 17, 457-474.

[4] Börger, R., Tholen, W. (1991). Strong, regular and dense generators. Cahiers de Topologie et Géométrie Différentielle Catégoriques, 32(3), 257–276.

[5] Bruyn, L. L. (2008). Noncommutative geometry and dual coalgebras. arXiv [Math.RA].

[6] Stenström, B. (1975). Rings of quotients: an introduction to methods of ring theory. Berlin Heidelberg: Springer.

[7] Kontsevich, M., Soibelman, Y. (2009). Notes on $A_\infty - algebras, A_\infty$-categories and non-commutative geometry. In Lecture Notes in Phys.: Vol. 757. Homological mirror symmetry (pp. 153–219).

[8] Reyes, M. L. (2012). Obstructing extensions of the functor Spec to noncommutative rings. Israel Journal of Mathematics, 192(2), 667–698.

[9] Cruttwell, G. (2008). Normed Spaces and the Change of Base for Enriched Categories. Dalhousie University.

[10] Bunge, M. C. (1969). Relative functor categories and categories of algebras. Journal of Algebra, 11(1), 64–101.

[11] Larson, R. E., Andima, S. J. (1975). The lattice of topologies: a survey. The Rocky Mountain Journal of Mathematics, 5(2), 177–198.

[12] Kelly, G. M. (1974). Doctrinal adjunction. In Gregory M. Kelly (Ed.), Category Seminar (pp. 257–280). Berlin, Heidelberg: Springer Berlin Heidelberg.

[13] Coulembier, K. (2023). Additive Grothendieck Pretopologies and Presentations of Tensor Categories. Applied Categorical Structures, 31(3), 23.

Ultracompletions

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Abstract. The notion of ultracategory was introduced by Michael Makkai in [8] for the characterisation of categories of models of pretoposes, an ample extension to (intuitionistic) first order theories of Stone duality for Boolean algebras, providing a kind of Stone duality for first order theories—aka cooonceptual completeness. Recently, Jacob Lurie refined that notion in [7] producing another approach to the duality for pretoposes—the two notions of ultracategory appear to be different, though no separating example has been produced yet. All this suggests that there are already two forms of duality for first order theories, in line with Esakia's duality as well as others, see [4, 2, 1].

A excellent, radically new, approach to ultrafilters, ultraproducts, ultraactegories, and pretoposes can be found in [5] where the author also foresees a possible comparison of the two original notions of ultracategories in future work.

In this work, we introduce a colax idempotent pseudomonad on an ***ultracompletion*** 2-functor on the 2-category **Cat** of small categories. Given a (small) category $\mathcal{C}$, write $\mathfrak{U}(\mathcal{C})$ for the category which consists of following data:

Objects are triples $(I,\mathcal{U},(c_i)_{i\in I})$ where $\mathcal{U}$ is an ultrafilter on the set I, and $(c_i)_{i\in I}$ is an I-indexed family of objects in $\mathcal{C}$.

An arrow $[V,f,(g_v)_{v\in V}]\colon(I,\mathcal{U},(c_i)_{i\in I})\to(J,\mathcal{V},(d_j)_{j\in J})$ is represented by a triple of a set $V\in\mathcal{V}$, a function $f\colon V\to I$ such that the inverse image of a set in $\mathcal{U}$ is a set in $\mathcal{V}$[1], and a family $(g_v\colon c_{f(v)}\to d_v)_{v\in V}$ of arrows in $\mathcal{C}$. Two representatives $(U,f,(g_v)_{v\in V})$ and $(U',f',(g'_v)_{v\in V'})$ are equivalent if $g_v=g'_v$ for all $v\in V\cap V'$.

Composition of arrows is given componentwise.

Remark. Let $\mathcal{T}$ denote a terminal category. The ultracompletion $\mathfrak{U}(\mathcal{T})$ is (equivalent to) the opposite of the category $\mathcal{UF}$ of ultrafilters of [5]. More generally, $\mathfrak{U}(\mathcal{C})$ is equivalent to $\left(\mathcal{UF}_{\mathrm{Fam}(\mathcal{C}^{\mathrm{op}})}\right)^{\mathrm{op}}$, where Fam is the usual coproduct completion of a category.

The assignment $\mathcal{C}\longmapsto\mathfrak{U}(\mathcal{C})$ extends to a 2-functor $\mathfrak{U}\colon\mathbf{Cat}\longrightarrow\mathbf{Cat}$, which we call ***ultracompletion***.

We briefly introduce the rest of the structure on the ultracompletion functor (write T for a fixed one-element set): for a fixed category $\mathcal{C}$, the unit functor $\nu_{\mathcal{C}}\colon\mathcal{C}\longrightarrow\mathfrak{U}(\mathcal{C})$ takes an object c to the triple $(T,\{T\},(c))$ consisting of a one-object family. The multiplication functor

$$\begin{array}{ccc}\mathfrak{U}(\mathfrak{U}(\mathcal{C})) & \xrightarrow{\qquad\mu_{\mathcal{C}}\qquad} & \mathfrak{U}(\mathcal{C})\\ (I,\mathscr{U},(J_i,\mathscr{V}_i,(c_j)_{j\in J_i})_{i\in I}) & \longmapsto & (\sum_{i\in I}J_i,\sum_{\mathscr{U}}\mathscr{V}_i,(c_{(i,j)})_{(i,j)\in\sum_{i\in I}J_i})\end{array}$$

[1] In other words, $f^{-1}\colon\wp(I)\to\wp(J)$ maps $\mathcal{U}\subseteq\wp(I)$ into $\mathcal{V}\subseteq\wp(J)$, see [5].

which employs the indexed sum of ultrafilters, see [5]. It is easy to see that they provide the data for a pseudomonad U on $\mathcal{C}at$. Finally we introduce a natural family of natural transformations

$$
\begin{tikzcd}
& & & (I,\mathcal{U},((T,\mathcal{T},(c_i)))_{i\in I}) \\
(I,\mathcal{U},(c_i)_{i\in I}) \arrow[urrr, bend left] \arrow[drrr, bend right] & \mathfrak{U}(\mathcal{C}) \arrow[r, shift left=2, "\mathfrak{U}(\nu_{\mathcal{C}})"] \arrow[r, shift right=2, "\nu_{\mathfrak{U}(\mathcal{C})}"'] \arrow[r, phantom, "\lambda_{\mathcal{C}}\Uparrow"] & \mathfrak{U}(\mathfrak{U}(\mathcal{C})) & \\
& & & (T,\mathcal{T},(I,\mathcal{U},(c_i)_{i\in I})) \arrow[uu, "{[I,!,[T,\mathrm{k}_i,(\mathrm{id}_{c_i})]_{i\in I}]}"']
\end{tikzcd}
$$

where $\mathrm{k}_i\colon T \to I$ is the constant function with value i.

Theorem. *The quadruple* $\mathrm{U} := (\mathfrak{U}, \mu, \nu, \lambda)$ *is a colax idempotent pseudomonad on* **Cat**.

The ultracompletion functor can be connected with both notions of ultracategories. For the sake of clarity, we shall denote by M-**Ultcat**, the 2-category of ultracategories, ultrafunctors, and natural ultra-transformations in the sense of Makkai's [8], and by L-**Ultcat**, the 2-category of ultracategories, ultrafunctors, and natural ultra-transformations in the sense of Lurie's [7].

Proposition. *Let* $\mathcal{C}$ *be a category.*
(i) The category $\mathfrak{U}(\mathcal{C})$ *is an ultracategory in the sense of Makkai, and the 2-functor* $\mathfrak{U}\colon \mathbf{Cat} \longrightarrow \mathbf{Cat}$ *factors through the forgetful 2-functor* M-$\mathbf{Ultcat} \longrightarrow \mathbf{Cat}$.
(ii) The category $\mathfrak{U}(\mathcal{C})$ *is an ultracategory in the sense of Lurie, and the 2-functor* $\mathfrak{U}\colon \mathbf{Cat} \longrightarrow \mathbf{Cat}$ *factors through the forgetful 2-functor* L-$\mathbf{Ultcat} \longrightarrow \mathbf{Cat}$.

Corollary. *Each* U*-pseudoalgebra* $\mathfrak{U}(\mathcal{C}) \xrightarrow{\alpha} \mathcal{C}$ *bears a structure of ultracategory in the sense of Makkai, and a structure of ultracategory in the sense of Lurie, in such ways that each assignment extends to a faithful 2-functor from* U-**PsAlg** *into* M-**Ultcat** *and into* L-**Ultcat**, *respectively.*

Finally, we have a result along the lines of Theorem 4.1 of [8].

Theorem. *Let* $\mathcal{P}$ *be a pretopos. The evaluation functor* $\mathrm{ev}\colon \mathcal{P} \longrightarrow \mathrm{U}(\mathrm{PreTop}(\mathcal{P}, \mathbf{Set}), \mathbf{Set})$ *is an equivalence of categories.*

The next steps will consider more closely the relationship between U-pseudoalgebras and ultracategories in the sense of Makkai, the connections with the work of Garner's in [5], and the abstract part of duality in line with previous work as in [3, 6, 9].

References

[1] J. Adámek and L. Sousa, *D-ultrafilters and their monads*, Adv. Math. 377 (2021), Paper No. 107486, 41.

[2] G. Bezhanishvili, N. Bezhanishvili, D. Gabelaia, and A. Kurz, *Bitopological duality for distributive lattices and Heyting algebras*, Math. Structures Comput. Sci. 20 (2010), no. 3, 359–393.

[3] G. D. Dimov and W. Tholen, *A characterization of representable dualities*, in *Categorical topology and its relation to analysis, algebra and combinatorics* (*Prague, 1988*), pages 336–357. World Sci. Publ., Teaneck, NJ, 1989.

[4] L. L. Esakia, *Topological Kripke models*, Soviet Math. Dokl. 15 (1974), 147–151.

[5] R. Garner. *Ultrafilters, finite coproducts and locally connected classifying toposes*, Ann. Pure Appl. Logic 171 (2020), no. 10, 102831, 29.

[6] P. T. Johnstone, *Stone Spaces*, Cambridge University Press, 1982.

[7] J. Lurie, Ultracategories, available at `www.math.ias.edu/~lurie/papers/Conceptual.pdf`, 2019.

[8] M. Makkai, *Stone duality for first order logic*, Adv. Math. 65 (1987), no. 2, 97–170.

[9] H.-E. Porst and W. Tholen, *Concrete dualities*, in *Category theory at work* (*Bremen, 1990*), volume 18 of *Res. Exp. Math.*, pages 111–136. Heldermann, Berlin, 1991.

Formal category theory via ∞-categorical proarrow equipments

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Abstract. The language of Joyal and Lurie's ∞-categories has become an indispensable tool in homotopy theory nowadays. However, some homotopical constructions are better phrased using other flavors of ∞-categories. For instance, it turns out that it is more convenient to work with ∞-categories internal to an ∞-topos of equivariant spaces in the field of equivariant homotopy, while it is essential to work with enriched ∞-categories in other contexts. It would thus be useful to have an overarching framework that produces theories of these types of generalized ∞-categories.

In this talk, we present an extension of methods from *formal* or *synthetic category theory* to the realm of ∞-categories. The field of formal category theory aims to distill the concepts enabling the formulation of well-behaved category theories internal to an ambient 2-category, with the prototypical example being the 2-category of categories. Building upon the foundational work of Street and Walters [4], Wood [1] developed a notion of *proarrow equipments*. This is an axiomatization of structure on a 2-category that allows one to define a good internal notion of pointwise Kan extensions, for instance.

By adopting Shulman's [3] and Verity's [2] double categorical perspective on these equipments, we will see that the theory of equipments naturally extends to the ∞-categorical context. An ∞-categorical equipment gives rise to well-behaved categorical concepts in its underlying $(\infty, 2)$-category such as Kan extensions, exact squares, and - under good conditions - notions of fibrations and comprehensive factorizations. There exist suitable equipments that yield the theory of indexed ∞-categories, more generally, ∞-categories internal to an ∞-topos, enriched ∞-categories, fibered ∞-categories and variants of these. Even more general, equipments that produce internal (∞, n)-category theories, may be constructed. We would like to highlight a few of these examples.

References

[1] R.J. Wood, *Abstract proarrows I*, Cah. Topol. Géom. Différ. Catég. **23** (1982), no. 3, 279–290.

[2] D. Verity, *Enriched categories, internal categories and change of base*, Ph.D. thesis, University of Cambridge, 1992.

[3] M. Shulman, *Framed bicategories and monoidal fibrations*, Theory Appl. Categ. **20** (2008), no. 18, 650-–738.

[4] R. Street and R. Walters, *Yoneda structures on 2-categories*, Journal of Algebra **50** (1978), no. 2, 350-–379.

A categorical view on signatures for inductive types

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Abstract.

Higher inductive-inductive types form a very general class of inductive types that seems to suffice to define all higher inductive types considered in the Homotopy Type Theory book [4], including the Cauchy reals. In [1] Kaposi and Kovács describe a way to specify higher inductive-inductive types (HIIT) in type theory. Kaposi and Kovács specify HIIT's using a *theory of signatures*: a specific type theory where contexts are signatures that encode a particular HIIT. We can morally see a signature for an inductive type as the presentation of some algebraic theory, and a theory of signatures as a class of algebraic theories.

In this talk, we will present a categorical analysis of these theories of signatures. We will first analyse the prototypical example of closed finitary inductive-inductive types, which correspond to finite presentations of generalized algebraic theories without equations. Looking at the theory of signatures we recover a strict version of Uemura's *representable map category* for the universal exponentiable arrow [3]. This motivating case of closed finitary inductive-inductive can then be extended; by enriching the theory of signatures on one side we get an enhanced notion of representable map category on the other side.

For example, going from closed signatures to open signatures corresponds to a fibered notion of representable map categories. More interestingly from the homotopical point of view, extending the signature from inductive-inductive to *higher* inductive-inductive corresponds to taking ∞-representable map categories [2].

Such a theory of signatures gives a syntactic specification for inductive types, which we can then interpret in any suitable target type theory. This should allow us to compare the strength of signatures and give an analysis of decompositions of signatures into basic type formers.

This is partially based on joint work with Evan Cavallo and Peter LeFanu Lumsdaine.

References

[1] A. Kaposi and A. Kovács. "Signatures and Induction Principles for HigherInductive-Inductive Types". In: *Logical Methods in Computer Science* 16 (2020). DOI: 10.23638/LMCS-16(1:10)2020

[2] Hoang Kim Nguyen and Taichi Uemura. "∞-type theories". arxiv:2205.00798. 2022.

[3] Taïchi Uemura. "A general framework for the semantics of type theory". In: *Mathematical Structures in Computer Science* 33 (2023). DOI: 10.1017/S0960129523000208

[4] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: https://homotopytypetheory.org/book, 2013.

The derivator associated to a dg-category

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Abstract.

Derivators, introduced independently by Grothendieck, Heller, Franke and further developed by Groth (see [4], [3]), yield a model of higher categories based on the language of 2-categories. A *prederivator* is a 2-functor

$$\mathbb{D}\colon \mathbf{Cat}^{\mathrm{op}} \to \mathbf{CAT}, \qquad I \mapsto \mathbb{D}(I).$$

and a *derivator* is a prederivator with additional properties. Here, we denote by **Cat** the 2-category of small categories and by **CAT** the 2-category of large categories – we also disregard logic and "size" issues here.

Heuristically, a derivator can be viewed as a collection of "homotopy categories of diagrams". A typical example is the *derivator associated to an ∞-category $\mathscr{C}$*, defined as follows:

$$\mathbb{D}_{\mathscr{C}}(I) = h(\mathscr{C}^I), \tag{0.1}$$

where $\mathscr{C}^I$ denotes the ∞-category of functors from (the nerve of) I to $\mathscr{C}$ and $h(-)$ denotes the homotopy category. Notice that, by taking $I = e$ the terminal category, we have $\mathbb{D}(e) = h(\mathscr{C})$. The homotopy category of $\mathscr{C}$ itself does not contain enough information to reconstruct $\mathscr{C}$, but the derivator $\mathbb{D}_{\mathscr{C}}$ does, in some sense (see for instance [2]). The properties we require of derivators essentially allow us to define *homotopy Kan extensions* and in particular *homotopy limits and colimits*. This is crucial in any version of higher category theory.

Higher categories, and hence also derivators, have a wide range of applications. In particular, they naturally appear in *homotopical and homological algebra*. If A is a ring, we may investigate its properties by introducing its *derived category* $\mathsf{D}(A)$. $\mathsf{D}(A)$ is defined as the localization of the category of chain complexes of A-modules along quasi-isomorphisms, namely, morphisms inducing isomorphisms in cohomology. The derived category $\mathsf{D}(A)$ is a *triangulated category*, which means that it has an additional structure allowing us to compute some homotopy limits and colimits in the form of *mapping cones*. Unfortunately, not all homotopy limits and colimits can be computed inside a triangulated category, and – even worse – they are *not functorial*.

What is true is that triangulated categories are almost always homotopy categories of higher categories; such higher categories, called *enhancements*, have the additional properties of being *stable*, which means that they in some sense behave like abelian categories. We have a theory of *stable ∞-categories* and, not surprisingly, a theory of *stable derivators* [3]. If $\mathscr{C}$ is a stable ∞-category, the

derivator $\mathbb{D}_{\mathscr{C}}$ (cf. (0.1)) is indeed a stable derivator. The derived category $\mathsf{D}(A)$ has a natural stable ∞-categorical enhancement, so it has also a stable derivator enhancement.

One might be content with the above picture, but there is catch. For a triangulated category, the most natural higher categorical enhancement is not described as a stable ∞-category or a stable derivator, but as a *differential graded (dg-) category*. A dg-category is a category *enriched in chain complexes*. In particular, if $\mathscr{A}$ is a dg-category, we may take its *homotopy category* $H^0(\mathscr{A})$ just by taking zeroth cohomology of hom complexes. If A is a ring, we may easily define its *derived dg-category* $\mathsf{D}_{\mathrm{dg}}(A)$, for which the equivalence $H^0(\mathsf{D}_{\mathrm{dg}}(A)) \cong \mathsf{D}(A)$ holds. Dg-categories are in fact yet another model of higher categories, one which is best suited for applications to homological algebra.

Inside a given dg-category $\mathscr{A}$, we can define well behaved *homotopy limits and colimits*, and *functorial mapping cones*. A dg-category having such mapping cones is called *pretriangulated*. This is the differential graded version of stability: the homotopy category $H^0(\mathscr{A})$ of a pretriangulated category $\mathscr{A}$ has a natural structure of triangulated category.

Now, we know that we can define a (pre)derivator associated to an ∞-category $\mathscr{C}$. If we start from a pretriangulated dg-category $\mathscr{A}$, we may take its *dg-nerve* $\mathrm{N}_{\mathrm{dg}}(\mathscr{A})$, which is a stable ∞-category, and then the associated (pre)derivator; still, in the existing literature there is no direct construction of a *(pre)derivator associated to a dg-category*. This work will close this gap. If $\mathscr{A}$ is a dg-category, we define:

$$\mathbb{D}_{\mathscr{A}}(I) = H^0(\mathbb{R}\mathrm{Hom}(I, \mathscr{A})), \tag{0.2}$$

where $\mathbb{R}\mathrm{Hom}(I, \mathscr{A})$ denotes the dg-category of *quasi-functors* [1] between the (free linear category generated by) I and $\mathscr{A}$. Quasi-functors are essentially "homotopically coherent functors" between dg-categories.

Our main result is that the above formula (0.2) yields a stable derivator, assuming that $\mathscr{A}$ is pretriangulated and *homotopy complete and cocomplete*. To show this, we develop a new theory of *homotopy limits and colimits* in dg-categories. We will also discuss some interesting applications to [5, Theorem 4.2] and to Gorenstein projective modules. In particular, thanks to the Ph.D. thesis of H.Holm, we know that the dg-category of totally acyclic complexes of projective modules is a dg-enhancement of the stable category of Gorenstein projective modules. For a small category I and a suitable ring R with finitely many objects, we can define an algebra RI. We aim to show that the derivator associated to the dg-category of totally acyclic complexes of projective R-modules, evaluated in I, is equivalent to the category of totally acyclic complexes of projective RI-modules.

References

[1] Alberto Canonaco and Paolo Stellari. Internal Homs via extensions of dg functors. *Adv. Math.*, 277:100–123, 2015.

[2] Daniel Fuentes-Keuthan, Magdalena Kędziorek, and Martina Rovelli. A model structure on prederivators for $(\infty, 1)$-categories. *Theory Appl. Categ.*, 34:Paper No. 39, 1220–1245, 2019.

[3] Moritz Groth. Derivators, pointed derivators and stable derivators. *Algebr. Geom. Topol.*, 13(1):313–374, 2013.

[4] Alexander Grothendieck. Les dérivateurs. Available online at https://webusers.imj-prg.fr/georges.maltsiniotis/groth/Derivateurs.html.

[5] Chiara Sava. ∞-dold-kan correspondence via representation theory. arXiv Preprint, 2022.

A 2-categorical model of oriented 1-cobordisms.

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Abstract.

The compact closed quasi category freely generated by one object is a much bigger object than the compact closed category generated by one object, in fact the latter is a mere 1-truncation of the former. Recall that the ∞-category of oriented 1-cobordisms $\mathbf{Bord}_1$ is the compact closed quasi-category freely generated by one object [AF21][H18][Lur09]. Our goal is to construct a symmetric monoidal bicategorical model of $\mathbf{Bord}_1$. Recall that the infinite loop space freely generated by a point $\Omega^\infty\Sigma^\infty(S^0)$ is the classifying space of a symmetric monoidal category Q constructed by Quillen in his proof of the Barrat-Priddy-Quillen theorem [Gra76]. We show that cospans in Q are naturally the 1-cells of a symmetric monoidal bicategory $CosQ$ [Sta16]. The bicategory $CosQ$ can be enlarged with the addition of 2-cells, called creation and a destruction operators. We conjecture that the resulting symmetric monoidal bicategory CCC is a model of the ∞-category $\mathbf{Bord}_1$ (in particular, $\mathbf{Bord}_1(A_0, A_1) \simeq \mathrm{B}CCC(A_0, A_1)$ for any pair of finite signed sets (A_0, A_1)). In support of the conjecture, we show that $CCC(\emptyset, \emptyset)$ is the symmetric monoidal category freely generated by Connes's cyclic category Λ and the space $\mathbf{Bord}_1(\emptyset, \emptyset)$ is the E_∞-space freely generated by $\mathbb{C}P^\infty$ (Recall that $\mathrm{B}\Lambda = \mathbb{C}P^\infty$ [DHK85]).

References

[AF21] D. Ayala and J. Francis, *Traces for factorisation homology in dimension 1*, arXiv:2105.01143v1, 2021.

[H18] Y. Harpaz, *The Cobordism Hypothesis in Dimension 1. 1*, 12100229v1, 2018.

[DHK85] W. G.. Dwyer, M. J. Hopkins, and D. M. Kan, *The homotopy theory of cyclic sets.*, Trans. American Math. Soc. (1985).

[Gra76] D. Grayson, *Higher algebraic k-theory: II (after D. Quillen)*, Lecture notes in Mathematics, 1976.

[Lur09] J. Lurie, *On the classification of topological field theories*, Current developments in mathematics, 2008, Int. Press, Somerville, MA, 2009, pp. 129–280.

[Sta16] M. Stay, *Compact closed bicategories*, Th. and Appl. of Categories **31** (2016), no. 26.

Toposes as standard universes

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Abstract.
E. Nelson introduced Internal Set Theory (IST) in 1977 [5], in an attempt to make the methods of nonstandard analysis more accessible to mathematicians (and physicists) not acquainted with Logic, particularly model theory. His approach was to extend ZFC by adding a new unary 'standardness' predicate $\mathrm{st}(x)$ and three axiom schemata (chiefly Transfer) to govern the behaviour of this new notion. This provides a reasonably contained set of rules one can use to make new proofs. The resulting theory is a conservative extension of ZFC, so that it raises no new foundational issues and can be used to prove classical results: any theorem of IST that can be stated in the language of ZFC is provable in ZFC, even if the IST proof cannot be expressed in ZFC.

There were several attempts to categorify nonstandard proof methods (e.g., [4, 7, 8]), but this talk follows a new perspective which is rather natural and addresses all three schemata from IST instead of focusing on just one in isolation. The point of view is that the additional axiom schemata of Internal Set Theory express relationships between hyperdoctrines (some of which are triposes [6]), envisioned as tools allowing us to abstract away from the ideas of internal formula, internal formula with standard parameters, and external formula while preserving their logical features. Starting from set theory as a template also allows us to leverage the well-known connections between topos theory and set theory [1, 2, 3, 9] — taking the "internal universe" to be an elementary topos with extra structure ought to be seen as a straight-up generalisation of starting from a model of ZFC with an additional predicate subject to some axioms.

Following [10], this talk will focus on toposes that closely resemble the set-theoretic models of IST by admitting a notion of 'standard element'. We will discuss the structure needed to implement transfer, standardisation, and idealisation internally to a topos, and sketch the proof that any topos satisfying the internal axiom of choice occurs as a universe of standard objects and maps [10]. This development allows one to employ these proof methods in environments such as toposes of G-sets and Boolean étendues.

References

[1] F. W. Lawvere. An elementary theory of the category of sets. Proceedings of the National Academy of Sciences of the United States of America, 52(6):1506 - 1511, 1964.

[2] W. Mitchell. Boolean topoi and the theory of sets. Journal of Pure and Applied Algebra, 2(3):261 - 274, 1972.

[3] G. Osius. Categorical set theory: A characterization of the category of sets. Journal of Pure and Applied Algebra, 4(1):79 - 119, 1974.

[4] A. Kock and C. J. Mikkelsen. Topos-theoretic factorization of non-standard extensions. In Victoria Symposium on Nonstandard Analysis, pages 122 - 143, Berlin, Heidelberg, 1974. Springer Berlin Heidelberg.

[5] E. Nelson. Internal set theory: A new approach to nonstandard analysis. Bull. Amer. Math. Soc., 83(6):1165 - 1198, 1977.

[6] J. M. E. Hyland, P. T. Johnstone, and A. M. Pitts. Tripos theory. Mathematical Proceedings of the Cambridge Philosophical Society, 88(2):205 - 232, 1980.

[7] I. Moerdijk. A model for intuitionistic non-standard arithmetic. Annals of Pure and Applied Logic, 73(1):37 - 51, 1995.

[8] E. Palmgren. Real numbers in the topos of sheaves over the category of filters. Journal of Pure and Applied Algebra, 160(2):275 - 284, 2001.

[9] M. Shulman. Comparing material and structural set theories. Annals of Pure and Applied Logic, 170(4):465 - 504, 2019

[10] J. Siqueira, Nonstandard proof methods in toposes, preprint arXiv:2308.16030v2, 2023. Accepted for publication in the Annals of Pure and Applied Logic.

Bicategories of algebras for relative pseudomonads

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Abstract.
Relative pseudomonads, introduced in [1], extend the notion of pseudomonad to non-endofunctors. We introduce pseudoalgebras for relative pseudomonads (showing that these recover the no-iteration algebras defined in [2]), define the bicategory of pseudoalgebras associated to a relative pseudomonad T, and establish its universal property amongst resolutions of T. We show that the Kleisli bicategory for T (constructed in [1]) embeds into the bicategory of T-pseudoalgebras as the full sub-bicategory of free T-pseudoalgebras; this provides a general coherence theorem when the codomain of T is a 2-category. As an application, we establish that the pseudoalgebras for the presheaf construction are the locally-small categories admitting small colimits.

References

[1] M. Fiore, N. Gambino, M. Hyland and G. Winskel, *Relative pseudomonads, Kleisli bicategories, and substitution monoidal structures*, Selecta Mathematica 24 (2018), no. 3, 2791–2830.

[2] F. Marmolejo and R. J. Wood, *No-iteration pseudomonads*, Theory and Applications of Categories 28 (2013), 371–402

Lax adjunctions and lax-idempotent pseudomonads

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Abstract.

Just as a family of U-universal maps $\eta_A : A \to UFA$ for a functor U gives rise to an adjunction, a family of maps satisfying certain "relative U-left Kan extension" conditions for a pseudofunctor U gives rise to a **lax adjunction**. We begin by presenting a generalization of Marta Bunge's result [1, Theorem 4.1] where this has been proven for 2-functors.

We then apply this result to the setting of **lax-idempotent pseudomonads** where we introduce a new technique for creating lax adjunctions out of biadjunctions. We give various examples, for instance we show that there is a canonical lax adjunction between the 2-category of algebras and the Kleisli 2-category for such pseudomonads. Another application guarantees a certain *enriched weak completeness* (in the sense of [3]) of the Kleisli 2-category. This for example applies to the bicategory PROF of locally small categories and "small" profunctors.

Finally, we show how (the dual of) this technique provides us with lax versions of classical results from two-dimensional monad theory, as described in [2, Section 5]. For instance, the 2-category of monoidal categories and lax monoidal functors, while not having many bicolimits, is weakly cocomplete in the above sense.

References

[1] M. Bunge, *Coherent extensions and relational algebras*, Transactions of the American Mathematical Society, 197 (1974): 355-390.

[2] R. Blackwell, G. M. Kelly, and A. J. Power, *Two-dimensional monad theory*, Journal of pure and applied algebra, 59.1 (1989): 1-41.

[3] S. Lack, J. Rosický, *Enriched weakness*, Journal of Pure and Applied Algebra, 216.8-9 (2012): 1807-1822.

Ordinals as Coalgebras: some missing categorical techniques

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Abstract.
Simply working out the characterisation of a standard categorical notion in a specific category often reproduces the textbook account of an old subject, or guides us in developing a new one.

Here we consider notions of "ordinal", using the theory of extensional well founded coalgebras, with the "down-sets" functor $\mathcal{D}$ on posets for the covariant powerset $\mathcal{P}$ in the original set-based example.

However, recovering the popular notion (a transitive extensional well founded relation) is not the easy application that we hoped for. It raises categorical questions that are simply stated and could be widely applicable but seem to be unknown.

Any binary relation $(\prec) \subset X \times X$ can be expressed as a coalgebra $\alpha : X \to \mathcal{P}X$. It is extensional iff α is mono and well-foundedness can be characterised by a "broken pullback" that I have discussed at categorical meetings. In our subject everything is up for negotiation: not only the category and the endofunctor but also the notion of "mono", which we replace with a factorisation system.

For the down-sets functor, a (well founded) coalgebra is a set with two binary relations $(X, \leq, \prec)$, where $(\leq)$ is a partial order, $(\prec)$ is a (well founded) binary relation and these are *compatible*,

$$z \leq y \prec x \Longrightarrow z \prec x \qquad \text{and} \qquad z \prec y \leq x \Longrightarrow z \prec x.$$

Then $f : Y \to X$ is a $\mathcal{D}$-*(coalgebra) homomorphism* iff

$$\begin{array}{llcl} \forall y y' : Y. & y' \leq_Y y & \Longrightarrow & f y' \leq_X f y \\ \forall y y' : Y. & y' \prec_Y y & \Longrightarrow & f y' \prec_X f y \\ \forall x : X. \forall y : Y. & x \prec_X f y & \Longrightarrow & \exists y' : Y. x \leq_X f y' \ \wedge \ y' \prec_Y y, \end{array}$$

whereas in the **Set** version we just had $x = f y'$, which we call a $\mathcal{P}$-*homomorphism*.

For the "monos", a categorist unencumbered by the historical baggage of set theory would use *lower sets*. This (easily) reproduces the (difficult) theory of *plump* ordinals in my 1996 JSL paper and has the universal property (*transfinite recursion*) with monotone successor that Joyal and Moerdijk identified in their contemporary *Algebraic Set Theory*. Plump ordinals grow very fast: $\omega \cdot 2$ does not exist in the simplest non-classical topos $\mathbf{Set}^{\to}$ without Replacement. You might think this situation is good or bad, but I intend to develop it into a purely categorical understanding of that logical principle.

For the popular notion we try using *regular* monos or full inclusions to redefine extensionality. Then $(\leq)$ is "set-theoretic inclusion" $(\subseteq)$ derived from $(\prec)$, which must be *meta*-transitive,

$$\forall w, x, y. \quad (\forall z. z \prec y \Rightarrow z \prec x) \ \wedge \ (x \prec w) \Longrightarrow (y \prec w).$$

The familiar *one-point successor* preserves this and there is a *rank* operation, *i.e.* a left adjoint to the inclusion amongst all well founded coalgebras. However, binary joins are very badly behaved and I don't know what transfinite recursion theorem might hold.

For an ordinary transitive $(\prec)$, its reflexive closure $(\preceq)$ is the appropriate choice for $(\leq)$, because then all $\mathcal{D}$-homomorphisms are actually $\mathcal{P}$-homomorphisms and lower inclusions. Binary joins (but with respect to $(\subseteq)$) are nicely behaved and transfinite recursion holds with inflationary successor.

But in developing the rank functor, we must consider (ordinary) extensionality and transitivity separately, falling back on symbolic methods and getting little benefit from known categorical theory.

We call a $\mathcal{D}$-coalgebra *transitive* if $\quad \forall xy.\ y \prec x \Longrightarrow y \leq x, \quad$ or $\alpha \leq \eta_X$.
This fits neatly with $(\mathcal{D}, \eta, \mu)$ being a Kock–Zöbelein monad, *i.e.* with $\mathcal{D}\eta_X \leq \eta_{\mathcal{D}X}$.

How can the transitive closure of general coalgebras be expressed 2-categorically?

It should be something like a *co-inserter*, but that's not it and, when I asked a senior 2-categorist, he didn't know what it was.

$$\begin{tikzcd}
\mathcal{D}X \arrow[rr, "\mathcal{D}f"] & & \mathcal{D}\overline{X} \\
X \arrow[u, shift right, "\alpha"] \arrow[u, shift left, "\eta_X"'] \arrow[rr, "f"] & & \overline{X} \arrow[u, shift right, "\overline{\alpha}"] \arrow[u, shift left, "\eta_{\overline{X}}"'] \arrow[u, phantom, "\leq"]
\end{tikzcd}$$

In the general theory, the extensional ("Mostowski") quotient is given by the fixed point of repeated factorisation of the structure map using the chosen notions of mono and epi. When the functor preserves the monos, this is actually just the longest corresponding epi.

However, $\mathcal{D}$ does *not* preserve plain monos, so to find the fixed point we need my notion of "well founded element". We call a regular epi homomorphism $g : A \twoheadrightarrow B$ *well projected* if it factors uniquely into every regular epi homomorphism $f : A \twoheadrightarrow E$ with E extensional:

$$\begin{tikzcd}
\mathcal{D}A \arrow[r, "\mathcal{D}g"] \arrow[rr, bend left, "\mathcal{D}f"] & \mathcal{D}B \arrow[r, dotted] & \mathcal{D}E \\
A \arrow[u, "\alpha"] \arrow[r, two heads, "g"] \arrow[rr, bend right, two heads, "f"'] & B \arrow[u, "\beta"'] \arrow[r, dotted] & E \arrow[u, tail, "\epsilon"']
\end{tikzcd}$$

How are well projected maps characterised order-theoretically?

So this piece of category theory has not fitted well with the traditional notion, but that could be the fault of the tradition.

In any case, constructively, the popular notion does not capture the more "combinatorial" ideas of ordinals used in subjects such as proof theory. That is because the functors $\mathcal{P}$ and $\mathcal{D}$ still use full higher order logic. But the reason for using category theory to generalise ideas is that quite different, maybe more combinatorial, functors could be used instead and would give entirely different results.

Relevant papers and seminar slides are at `www.PaulTaylor.EU/ordinals/`

Enriched universal algebra

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Abstract.
Universal algebra, introduced by Birkhoff, deals with sets A equipped with functions $f_A\colon A^n \to A$, where f is a function symbol of arity n in a certain *language* (or signature) $\mathbb{L}$. Starting from this, one builds *terms* and *equations*, and studies the classes of algebras satisfying a given set of equations.

A categorical treatment of universal algebra was given by Lawvere through the concept of *Lawvere theory*. This was further developed in the context of *enriched categories* by several authors; including Lack and Power, Bourke and Garner, and Lucyshin-Wright and Parker. Such generalizations follow the categorical approach of Lawvere, but do not yet provide a notion of *enriched universal algebra* with function symbols, recursively generated terms, and equations. In fact, instances of this have been developed only in specific situations: notably over posets, metric spaces, and complete partial orders.

In this talk, mostly based on a joint paper with Rosický, we unify this fragmented picture under the same general theory, providing new useful tools that will allow the development of universal algebra in other areas of enriched category theory.

⋆

Our starting point is a *language* $\mathbb{L}$ which consists of a set of (X, Y)-ary function symbols, whose input and output arities are objects of the base of enrichment $\mathcal{V}$. Then we define $\mathbb{L}$-*structures* and $\mathbb{L}$-*terms*; the latter are constructed recursively out of the function symbols of $\mathbb{L}$, the morphisms of $\mathcal{V}$, and by incorporating the monoidal closed structure of $\mathcal{V}$. Interpretation of terms in $\mathbb{L}$-structures comes next:

$$t \in (X, Y)\text{-ary term, } A \in \mathbb{L}\text{-structure} \quad \mapsto \quad t_A\colon A^X \to A^Y \text{ in } \mathcal{V}.$$

And finally, an *equational* $\mathbb{L}$-*theory* $\mathbb{E}$ is defined as a family of equations $\{s_j = t_j\}_{j\in J}$ between terms of the same arity; models of $\mathbb{E}$ are $\mathbb{L}$-structures satisfying the interpreted equations.

If the base category $\mathcal{V}$ is locally finitely presentable as a closed category, we can talk about *finitary* equational theories just by restricting the input arities to vary among the finitely presentable objects of $\mathcal{V}$. In the same vein, one can define Φ-ary equational theories for a sound class of weights Φ.

Then, our first result shows that $\mathcal{V}$-categories of models of finitary equational theories can be equivalently described as $\mathcal{V}$-categories of algebras for finitary enriched monads on $\mathcal{V}$, generalizing the ordinary results for finitary varieties. In the sound case, the $\mathcal{V}$-categories of models of Φ-ary equational theories are the $\mathcal{V}$-categories of algebras of Φ-ary enriched monads on $\mathcal{V}$.

Secondly, we determine the simplest set of output arities that are necessary to express models of equational theories. In particular we will see why in the case of $\mathcal{V} = \mathbf{Set}, \mathbf{Pos}, \mathbf{Met}$, and ω-$\mathbf{CPO}$ it is enough to consider terms with terminal output arity, and we will get hints on how to develop new applications, including for instance 2-categorical and simplicial universal algebra.

Cauchy convergence for normed categories

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Abstract. Building on the notion of normed category as suggested by Lawvere [1], we introduce notions of Cauchy convergence and cocompletenes for such categories which differ from proposals in previous works, such as [2]. Key to our approach is to treat them consequentially as categories enriched in the monoidal-closed category of normed sets, *i.e.*, of sets which come with a norm function. Our notions largely lead to the anticipated outcomes when considering individual metric spaces or normed groups as small normed categories (in fact, groupoids), but they can be quite challenging when trying to establish them for large categories, such as those of semi-normed or normed vector spaces – not just because norms of vectors need to be allowed to have value ∞ in order to guarantee the existence of colimits of (sufficiently many) infinite sequences.

The interesting large normed categories typically have objects with some quantitative structure which, however, gets largely ignored by their morphisms, such as normed vector spaces with all linear maps. But the object structure is then used to declare norms of morphisms which enable one to identify meaningful types amongst them, just as the usual operator norm identifies bounded or 1-Lipschitz operators of vector spaces. Working with a general commutative quantale $\mathcal{V}$, rather than only with Lawvere's quantale $\mathcal{R}_+$ of real numbers, we will demonstrate that the categorically atypical and, in fact, questionable structure gap between objects and morphisms is already visible in the underlying normed category of the enriching category of $\mathcal{V}$-normed sets. To show that this normed category and, in fact, all presheaf categories over it, are Cauchy cocomplete, we assume the quantale $\mathcal{V}$ to satisfy alternatively a couple of light extra properties which, however, are present in all instances of interest to us. Of utmost importance to the general theory is the fact that our notion of Cauchy convergence is subsumed by the notion of weighted colimit of enriched category theory. With this theory and, in particular, with a result of [3], we are able to prove that all $\mathcal{V}$-normed categories have Cauchy cocompletions, for $\mathcal{V}$ satisfying our alternative light assumptions.

(We emphasize that our notion of Cauchy cocompleteness is not to be confused with the selfdual idempotent-split property of a category, often referred to as Cauchy completeness. Time permitting, we will comment on the connection between the two notions. For all details on this and on any other facts and examples, we must refer to the forthcoming [4].)

References

[1] F.W. Lawvere, *Metric spaces, generalized logic, and closed categories*, Rend. Sem. Mat. Fis. Milano 43, 135–166, 1973. Republished in: Reprints in Theory Appl. Categories 1, 2002.

[2] W. Kubiś, *Categories with norms*, preprint, arXiv:1705.10189v1 [math.CT], 2017.

[3] G.M. Kelly and A. Schmitt, *Notes on enriched categories with colimits of some class*, Theory Appl. Categories 14, no. 17, 399-423.

[4] M.M. Clementino, D. Hofmann and W. Tholen, *Cauchy convergence in $\mathcal{V}$-normed categories*, in preparation.

A classifying localic category for locally compact locales

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Abstract. It is reasonably clear how to find a classifying localic groupoid for locally compact locales. That is, we can find a localic groupoid $\mathbb{G}$ such that principal $\mathbb{G}$-bundles over any locale X are equivalent to locally compact locales in the topos of sheaves, $Sh(X)$. This is like a localic version of having a classifying topos for geometric theories, but not quite as the morphisms between principal bundles are all isomorphisms. So really, all we are 'classifying' are locally compact locales with isomorphisms between them, not locale maps in general. This is in contrast to classifying toposes which classify the models of a geometric theory *and* their morphisms. Further it seems hard to avoid this problem since, as is well known, all morphisms between principal bundles are isomorphisms. In this talk I'll present a way round the issue by defining morphisms between principal bundles as certain principal bundles associated with the arrow category naturally arising in the construction of $\mathbb{G}$.

In fact the underlying results are quite general and can be summarised by a nice characterisation of geometric stacks of *categories* on any cartesian category $\mathcal{C}$. Here a geometric stack of *categories* is just a geometric stack of groupoids (i.e. $X \mapsto Prin_{\mathbb{G}}(X)$) but using the new notion of morphism between principal bundles to define the morphisms of each category of principal bundles $Prin_{\mathbb{G}}(X)$.

Operads colored by categories

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Abstract. Colored operads, also known as symmetric multicategories, have proven to be useful in various mathematical disciplines, such as algebraic topology, algebraic geometry, homotopy theory, or mathematical physics.

For a set C, a C-colored operad P consists of objects $P(c_1 \cdots c_n; c)$ of abstract n-ary operations, whose inputs and output have specific types $c_1, \dots, c_n, c \in C$, together with associative and unital composition maps. We consider a generalisation, where the colors form a category; the morphisms of the coloring category act on inputs and output of the operations, possibly changing their type. Such generalisation of colored operads originally appeared in [DS03] under the name 'symmetric substitudes', and later independently in [Petersen13, Ward22]. There are applications of category-colored operads in deformation theory [DSVV24] and in homotopy theory [BW22].

I will present a new definition based on partial compositions, which is suitable for non-unital version of operads. The idea is to replace the sources of the operad composition

$$P_n \otimes P_m \xrightarrow{\circ_i} P_{n+m}$$

by certain colimits $P_n \otimes_i P_m$, which deal with the categorical coloring. This approach leads us to realize that:

Result I: *Category-colored operads are internal algebras of a certain categorical operad of functors.*

In the homotopy theory of algebras, it is standard to encode an algebra as an algebra of an operad. If this operad is binary quadratic (meaning it is free modulo quadratic relations and generated by binary operations) and 'Koszul', there is an algorithm for finding its minimal cofibrant replacement. Algebras of this replacement are then homotopy versions of the original algebras [Markl04]. To imitate this process with operads in place of algebras, it was essential to pass to the category-colored setting, which resulted in:

Result IIa: *There is a quadratic binary non-unital operad, colored by a groupoid of permutations Σ, whose algebras are non-unital symmetric operads.* This directly extends to:

Result IIb: *There is a quadratic binary non-unital category-colored operad, whose algebras are non-unital Markl $\mathbb{O}$-operads for an operadic category $\mathbb{O}$.*

Operadic categories were introduced in [BM15] as a unifying framework for various operadic structures (such as cyclic and modular operads, di–operads, &c.) and their homotopy theory.

The **Results** can be found in my recent article [T23], on which the talk is based.

References

[BM15] M.A. Batanin and M. Markl. Operadic categories and duoidal Deligne's conjecture. *Advances in Mathematics*, 285:1630–1687, 2015.

[BW22] M.A. Batanin and D. White. Homotopy theory of algebras of substitudes and their localisation. *Transactions of the American Mathematical Society*, 375(05):3569–3640, 2022.

[DS03] B. Day and R. Street. Abstract substitution in enriched categories. *Journal of Pure and Applied Algebra*, 179(1-2):49–63, 2003.

[DSVV24] V. Dotsenko, S. Shadrin, A. Vaintrob, and B. Vallette. Deformation theory of cohomological field theories, *Journal für die reine und angewandte Mathematik (Crelles Journal)*, ahead of print, 2024.

[Markl04] M. Markl. Homotopy algebras are homotopy algebras. *Forum Mathematicum*, 16:129–160, 2004.

[Petersen13] D. Petersen. The operad structure of admissible G-covers. *Algebra & Number Theory*, 7(8):1953–1975, 2013.

[T23] D. Trnka. Category-colored operads, internal operads, and Markl O-operads, *Theory and Applications of Categories*, 39:874–915, 2023.

[Ward22] B. C. Ward. Massey Products for Graph Homology, *International Mathematics Research Notices*, 2022(11):8086-8161, 2022.

Existential completions, AC-chaotic situations and towers of toposes

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Abstract. In this talk, we introduce a new hierarchy of toposes obtained by combining the tripos-to-topos construction [1] with the full existential completion [5, 7]. Then, we compare this new tower of toposes with the one introduced by Menni in [3, 4], obtained by iterating the ex/lex and reg/lex completions. Menni proved that, under suitable hypothesis (AC-chaotic situation, weak dependent products and a generic object), a given lex category $\mathcal{C}$ provides a tower of toposes $(\mathcal{C}_{\mathsf{reg/lex}(n)})_{\mathsf{ex/lex}}$, and each topos in the hierarchy is a sheaf subtopos (for the canonical topology) of the next one. For example, when $\mathcal{C}$ is the category of partitioned assemblies $\mathsf{PAsm}(\mathcal{K}_1)$, the first step of the tower of toposes if the effective Eff topos, while the second one, i.e. $(\mathsf{PAsm}(\mathcal{K}_1)_{\mathsf{reg/lex}})_{\mathsf{ex/lex}}$ is the topos of extensional realizability studied by van Oosten in [6]. While when $\mathcal{C}$ is the quasi-topos obtained as coproduct completion H_+ of a frame H, then $((H_+)_{\mathsf{reg/lex}})_{\mathsf{ex/lex}}$ is equivalent to the topos $\mathsf{Set}^{(D_+^n H)^{op}}$, where D_+H denotes the frame of non-empty lower subsets of H. Menni conjectured that the hierarchy of realizability toposes can be presented in tripos-theoretic way, and Hofstra provides a formal answer to this problem in [2].
In detail, we present the following theorem:

Theorem 1. *Let $P\colon \mathcal{S}^{op} \to \mathsf{InfSl}$ be a primary doctrine on a topos whose epi splits and such that the Grothendieck category $\mathcal{G}_P$ has weak dependent products and a generic object with respect to the (canonical) AC-chaotic situation on $\mathcal{S}$. Then:*

- *every full existential completion $P^{\exists(n)}\colon \mathcal{S}^{op} \to \mathsf{Hey}$ is a tripos (for $n \geq 1$);*
- *every Grothendieck category $\mathcal{G}_{P^{\exists(n)}}$ has a (canonical) AC-chaotic situation on $\mathcal{S}$, weak dependent products, and a generic object;*
- *we have a new hierarchy of toposes obtanined as tripos-to-topos $\mathsf{T}_{P^{\exists(n)}}$, and each topos in this hierarchy is a sheaf subtopos of the next one for a topology abstracting the ordinary sheafification.*

Moreover, $(\mathcal{G}_P)_{\mathsf{ex/lex}} \equiv \mathsf{T}_{P^{\exists}}$ and for every $n \geq 1$ we have that the topos $((\mathcal{G}_P)_{\mathsf{reg/lex}(n)})_{\mathsf{ex/lex}}$ is a *reflective subcategory of the topos* $\mathsf{T}_{P^{\exists(n+1)}}$*, graphically:*

$$\begin{tikzcd}
\mathsf{T}_{P^{\exists}} \arrow[r, bend right, ""{name=A1}] \arrow[d, equal] & \mathsf{T}_{P^{\exists(2)}} \arrow[l, bend right] \arrow[r, phantom, "\perp"] \arrow[r, bend right] \arrow[d, bend right] \arrow[d, phantom, "\dashv"] & \mathsf{T}_{P^{\exists(3)}} \arrow[l, bend right] \arrow[r, bend right] \arrow[d, bend right] \arrow[d, phantom, "\dashv"] & \mathsf{T}_{P^{\exists(4)}} \ldots \arrow[l, bend right] \arrow[d, bend right] \arrow[d, phantom, "\dashv"] \\
(\mathcal{G}_P)_{\mathsf{ex/lex}} \arrow[r, bend right] & ((\mathcal{G}_P)_{\mathsf{reg/lex}})_{\mathsf{ex/lex}} \arrow[l, bend right] \arrow[u, bend right] \arrow[r, bend right] & ((\mathcal{G}_P)_{\mathsf{reg/lex(2)}})_{\mathsf{ex/lex}} \arrow[l, bend right] \arrow[u, bend right] \arrow[r, bend right] & ((\mathcal{G}_P)_{\mathsf{reg/lex(3)}})_{\mathsf{ex/lex}} \ldots \arrow[l, bend right] \arrow[u, bend right]
\end{tikzcd}$$

First, the proof of the existence of our new tower is achieved through the following result:

Proposition 2. *Let* $P\colon \mathcal{S}^{op} \to \mathsf{Hey}$ *be a tripos on a topos whose epi splits. Then the Grothendieck category* $\mathcal{G}_P$ *has a canonical AC-chaotic situation on* $\mathcal{S}$*, weak dependent products and a generic object. Hence, its full existential completion is a tripos.*

Then, the existence of the adjunctions between our tower of toposes and Menni's one follows by combining the universal properties of the completions involved with a new decomposition of the full existential completion, which allows us to provide a characterization of the regular completion of Grothendieck categories which generalizes the known equivalence $\mathsf{Asm}(\mathbb{A}) \equiv \mathsf{PAsm}(\mathbb{A})_{\mathsf{reg/lex}}$. In detail:

Proposition 3. *The full existential completion of doctrines whose base category is regular can be decomposed into two steps: by first adding left adjoints along the class of all the regular epi, and then by adding left adjoints along the class of monos. Moreover, for every primary doctrine* $P\colon \mathcal{C}^{op} \to \mathsf{InfSl}$ *with* $\mathcal{C}$ *regular category, we have that the category* $(\mathcal{G}_P)_{\mathsf{reg/lex}}$ *is equivalent to the category* $\mathcal{G}_{P^{\exists_{\mathsf{e}}}}$*, i.e. the Grothendieck category of the doctrine* $P^{\exists_{\mathsf{epi}}}$ *obtained by adding left adjoints along regular epis.*

We conclude by presenting relevant examples and applications of our main results, including all Set-based triposes. Finally, we will discuss the specific towers arising from realizability and localic triposes.

References

[1] J.M.E. Hyland, P.T. Johnstone, and A.M. Pitts. Tripos theory. *Math. Proc. Camb. Phil. Soc.*, 88:205–232, 1980.

[2] P. Hostra. *Completions in Realizability.* PhD thesis, Universiteit Utrecht 2003.

[3] M. Menni. *Exact Completions and Toposes.* PhD thesis, University of Edimburgh 2000.

[4] M. Menni. More exact completions that are toposes. *Annals of Pure and Applied Logic*, 116(1):187–203, 2002.

[5] M.E. Maietti and D. Trotta. A characterization of generalized existential completions. *Annals of Pure and Applied Logic*, 174(4):103234, 2023.

[6] J. van Oosten Extensional realizability. *Annals of Pure and Applied Logic*, 84(3): 317-349, 1997.

[7] D. Trotta. The existential completion. *Theory and Applications of Categories*, 35:1576–1607, 2020.

Recent progress in the theory of effective Kan fibrations in simplicial sets

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Abstract.
One of the most important steps in the development of homotopy type theory has been the construction, by Voevodsky, of a model of type theory with the univalence axiom in the category of simplicial sets [7]. This work builds on the classic Kan-Quillen model structure in simplicial sets.

From the very beginning people have been trying to understand how *constructive* Voevodsky's results are. Besides being of intrinsic interest, it is also relevant for the question whether these results hold relative to an arbitrary base topos. Perhaps most importantly it also asks whether one can *compute* with the univalence axiom, or any other principle that might hold in the simplicial model.

Early on, an obstruction was found by Bezem, Coquand and Parmann [4]. They observed that the classic result saying that the exponential A^B is a Kan complex whenever A and B are, is not provable constructively. We refer to this as the *BCP-obstruction.* Since Kan complexes are interpreting the types in Voevodsky's model and the exponentials are the obvious way to interpret function spaces, this blocks a direct constructive interpretation of function types in Voevodsky's model.

In response most researchers have switched to *cubical sets.* This does not only involve changing the shapes, but also involves strengthening the notion of a Kan comples, or a Kan fibration, by adding *uniformity conditions* [3, 5]. Indeed, the usual definition of a Kan complex requires the mere existence of fillers against a class of maps, whether these are horn inclusions or open box inclusions. The other innovation is to insist that a Kan fibration comes equipped with a system of solutions which is required to satisfy certain compatibility conditions. It is in this way that one can overcome the BCP-obstruction in cubical sets.

While this has sometimes been taken to mean that cubical sets are constructively superior, matters are really not that clear. Indeed, as observed by Gambino and Sattler [6], uniformity conditions can also be used to overcome the BCP-obstruction in simplicial sets. Indeed, in their paper they define a notion of a uniform Kan complex, mirroring the cubical definition, and show that if A is a uniform Kan complex, then so is A^B for any simplicial set B.

In a book written with Eric Faber [1], we gave another solution which we call *effective Kan fibrations*, using uniformity conditions stronger than those of Gambino and Sattler. In contrast to Gambino and Sattler's notion, our definition is *local.* This means that we can show the existence of universal effective Kan fibrations, which should allow us to interpret type-theoretic universes. Indeed, the main results of our book are:

(1) Every effective Kan fibration is a Kan fibration in the usual sense, and in a classical metatheory

one can show that every Kan fibration can be equipped with the structure of an effective Kan fibration.

(2) Whenever f and g are effective Kan fibrations, then so is $\Pi_f(g)$, the push forward of g along f.

(3) Being an effective Kan fibration is a local property and hence universal effective Kan fibrations exist.

The ultimate aim is to develop a constructive proof of the existence of both a model of homotopy type theory and the Kan-Quillen model structure on simplicial sets using the notion of an effective Kan fibrations. Unfortunately, this remains work in progress.

In the meantime, the speaker has obtained, often together with (former) MSc students, some further results and the purpose of this talk is to report on these. In particular, we have shown that:

1. Any simplicial group is effectively Kan ([2], jww with Freek Geerligs).
2. The effective Kan fibrations are cofibrantly generated by a countable double category ([2], jww with Freek Geerligs). Classically, this means they are the right class in algebraic weak factorisation system.
3. Whenever f is an effective Kan fibration, then W_f, the W-type associated to f is an effective Kan complex (jww with Shinichiro Tanaka).
4. A version of the Joyal-Tierney calculus works for effective Kan fibrations (jww with Eric Faber).

Since we are still working on related issues, we may have more to report in June.

References

[1] Benno van den Berg and Eric Faber. *Effective Kan fibrations in simplicial sets*. Lecture Notes in Mathematics, volume 2321, Springer, Cham, 2022.

[2] Benno van den Berg and Freek Geerligs. *Examples and cofibrant generation of effective Kan fibrations*. arXiv:2402.10568, 2024.

[3] Marc Bezem, Thierry Coquand and Simon Huber. *The Univalence Axiom in Cubical Sets*, J. Autom. Reasoning, volume 63(2) (2019), pp. 159–171.

[4] Marc Bezem, Thierry Coquand and Erik Parmann. *Non-constructivity in Kan simplicial sets*, 13th International Conference on Typed Lambda Calculi and Applications,LIPIcs. Leibniz Int. Proc. Inform., volume 37 (2015), pp. 92–106.

[5] Cyril Cohen, Thierry Coquand, Simon Huber and Anders Mörtberg. *Cubical Type Theory: A Constructive Interpretation of the Univalence Axiom*, FLAP, volume 4 (10), 2017, pp. 3127–3170.

[6] Nicola Gambino and Christian Sattler. *The Frobenius condition, right properness, and uniform fibrations*, J. Pure Appl. Algebra, volume 221 (12) 2017, pp. 3027–3068.

[7] Krzysztof Kapulkin and Peter LeFanu Lumsdaine. *The simplicial model of univalent foundations (after Voevodsky)*, J. Eur. Math. Soc. (JEMS), volume 23 (6), 2021, pp. 2071–2126.

Associativity of Cosmash Products in Non-associative Algebras

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Abstract.
The purpose of this talk is to convince you that, for a field K with characteristic zero, the condition called *cosmash associativity* characterizes the variety of commutative and associative K-algebras amongst all varieties of non-associative K-algebras. Since this condition is motivated by Category Theory, this gives a very abstract way of thinking about commutativity and associativity.

In order to present things in an understandable way, we will first recall the notion of the *binary cosmash product*. We see how it naturally leads to a suitable definition of binary commutators by looking at some classical examples. In the case of commutative associative algebras, it corresponds to the binary tensor product. We then try to extend these notions to the ternary case, and even to the n-ary case for some natural number n. From this point, what *cosmash associativity* means can be explained essentially without effort. In the end, we discuss the main result and the techniques used to prove it. If time allows it, we mention new questions which appeared from this project.

Joint work with Ülo Reimaa and Tim Van der Linden.

References

[1] A. Carboni, G. Janelidze,*Smash products of pointed objects in lextensive categories*, Journal of Pure and Applied Algebra 183 (2003),

[2] M. Hartl, T. Van der Linden, *The ternary commutator obstruction for internal crossed modules*, Adv. Math. 232 (2013)

[3] S. Mantovani, G. Metere, *Normalities and Commutators*, J. Algebra 234 (2010)

[4] Ü. Reimaa, T. Van der Linden, C. Vienne, *Associativity and the Cosmash Product in Operadic Varieties of Algebras*, Illinois J. Math 67 (3) 563-598 (2023).

Constructing generalized schemes using cone injectivity

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Abstract.

We provide generalizations of two notions of a scheme from algebraic geometry. Both are defined as certain geometric objects admitting an open cover by "affine schemes". In the first approach, affine schemes constitute the image of a spectrum functor valued in a suitable category generalizing that of locally ringed spaces. The spectrum functor is constructed using cone injectivity and the construction works quite generally, especially for a locally finitely presentable category $\mathcal{A}$. In this full generality however, the spectrum functor fails to be fully faithful and we explain reasonable sufficient conditions under which it is. In the second approach, we develop a generalization of another concept from algebraic geometry – the functor of points, valued in a certain category of (small) sheaves on $\mathcal{A}^{\mathrm{op}}$. Finally, assuming the full faithfulness of the spectrum functor, we prove equivalence of the two resulting notions of schemes. On the way, we prove a useful universal property of the category of small sheaves.

This is based on a joint work with J. Jurka and T. Perutka, written up in an arxiv preprint [1].

References

[1] J. Jurka, T. Perutka, L. Vokřínek, Constructing generalized schemes using cone injectivity, preprint, arXiv:2201.03516v3, 2023.

Formal theory of Rezk completions

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Abstract.

Our purpose is to illuminate and to extend the theory of *univalent categories* in homotopy type theory. To this end, we generalize from the bicategory Cat and univalent categories therein to "Cat-like" 2-categories, equipped with a Yoneda structure, and univalent objects therein. In particular, we generalize and refine the result that weak equivalences (between univalent categories), are necessarily isomorphisms (adjoint equivalences). Consequently, we conclude that any fully faithful and (mere) essentially surjective functor induces an isomorphism between the univalent completions (i.e., the Rezk completion) of the source and target object.

Univalent category theory. In univalent foundations, every mathematical object comes equipped with its notion of sameness, and reasoning is invariant under this notion. The univalence axiom implies that univalent categories are necessarily invariant under weak equivalences. Therefore, the most useful notion of category is that of univalent category, in univalent foundations. However, not every *classical* construction is closed under univalence, such as Kleisli categories constructed via Kleisli morphisms. That is, the Kleisli category (as before) is not-necessarily univalent, even if the underlying category is univalent. A process to turn a non-univalent category into a univalent one is provided in [2] ; it provides a construction of the "free univalent" category, referred to as the *Rezk completion*. Concretely, the Rezk completion of a category is constructed as the full subcategory of *representable* presheaves. The universal property of the Rezk completion [2] , says precisely that unit of associated adjunction (given by the inclusion $\mathsf{Cat}_{\mathsf{univ}} \hookrightarrow \mathsf{Cat}$) is a pointwise weak equivalence. The pointwise weak equivalences are shown to correspond with the fully faithful and essentially surjective functors. (the latter are referred to as the *weak equivalences* of categories.)

Our work. In this abstract, we generalize the results concerning the theory of univalent categories, weak equivalences and Rezk/univalent completions, see [2, 5]. To achieve this generalization we build upon an existing theory of (univalent) bicategories [4] . Additionally, we formulate a type theoretic version of Yoneda structures [1] . This study requires to reason about categories modulo isomorphism, instead of equivalence. Hence, we consider those bicategories equipped with an "underlying precategory", referred to as 2-categories. We universally characterize the essentially surjective functors relative to a Yoneda structure, generalizing the proof that the weak equivalences of categories are the pointwise weak equivalences.

Yoneda structures. From now on, we fix a 2-category $\mathcal{K}$ equipped with a Yoneda structure, see [1] . (Informally, the objects of $\mathcal{K}$ are to be interpreted as (V-enriched) 1-categories.) The Yoneda structure on $\mathcal{K}$ assigns to every object X an object $\mathcal{P}\,X$ (its object of presheaves) and a morphism $よ_X : X \to \mathcal{P}\,X$ (its Yoneda morphism), see [1] for the universal property of $(X,\ よ_X)$.

The main idea behind a Yoneda structure is that every morphism is uniquely determined by its action on "generalized objects" and "generalized morphisms" respectively. The idea is made formal by the following construction, due to Street and Walters.

Construction. Every precomposition functor $\mathcal{K}(f, Z)$ factors through a displayed category over the source (hom-)category, denoted $ExNat(f, Z)$:

$$\begin{tikzcd} & ExNat(f,Z) \arrow[d, "\pi_1"] \\ \mathcal{K}(Y,Z) \arrow[ur, dashed, "(f\cdot^{e}_{Z} -)"] \arrow[r, "(f\cdot -)"'] & \mathcal{K}(X,Z) \end{tikzcd}$$

Definitions. Let X, Y, Z be objects and $f : X \to Y$ a morphism. Then f is *essentially surjective* if $(f\cdot^{e}_{Z})$ is a weak equivalence of (univalent) categories, for every univalent Z; where an object Z is *univalent* if for any $f : X \to Y$, the category $ExNat(f, Z)$ is univalent.

Theorem. Let $f : X \to Y$ be a morphism. Assume f is a *weak equivalence*, i.e., f fully faithful [1] and essentially surjective. Then, for every univalent Z, the precomposition functor $\mathcal{K}(f, Z)$ is an isomorphism of (hom-)categories. (If f satisfies the latter, f is said to be a "local equivalence".) The converse holds if for every object, a weak equivalence into a univalent object is given.

Example. The motivating type of 2-categories are those of the form $\mathcal{K} := \mathsf{Cat}_{\mathcal{V}}$, 2-categories of $\mathcal{V}$-enriched categories ($\mathcal{V}$-functors, and $\mathcal{V}$-transformations). $\mathcal{V}$ is assumed to be strong enough as a base for enrichment. Furthermore, we require closedness and completeness in order to construct the Yoneda structure on $\mathcal{V}$.

The theorem means precisely that the "local equivalences" are given by those morphisms which are fully faithful (intuitively, "inclusions" of hom-categories) and essentially surjective on (a subtype of) objects. Consequently, the univalent objects and weak equivalences are suitably characterized internal to $\mathcal{K} = \mathsf{Cat}_{\mathcal{V}}$. Furthermore, the Rezk completion and can be constructed as the (replete) (eso,ff)-image of its Yoneda morphism, i.e., as a full subcategory of its presheaf category.

Conclusion. In this project, we provide an axiomatic framework generalizing the concrete construction of [2, 5] to a more abstract (already existing) setting: 2-categories equipped with a Yoneda structure [1]. We observe that a Yoneda structure on a 2-category provides sufficient structure to suitably interpret weak equivalences as morphisms which are essentially surjective on objects and fully faithful. This interpretation is based on (a slight generalization of) Proposition 23 in [1]. The theory presented in the framework, axiomatizes the structure a 2-category needs to have, in order to suitably construct, and reason, without worrying about "univalence-requirements".

References

[1] R. Street, R. Walters, *Yoneda structures on 2-categories*, second ed., Journal of Algebra, vol. 50,Academic Presss, 1978.

[2] B. Ahrens, K. Kapulkin, M. Shulman, *Univalent categories and the Rezk completion*, fifth ed., Mathematical Structures in Computer Science, vol. 25, Cambridge University Press, 2015.

[3] The Univalent Foundations Program, *Homotopy Type Theory: Univalent Foundations of Mathematics*, https://homotopytypetheory.org/book, 2013.

[4] A. Benedikt and D. Frumin, M. Maggesi, N. Veltri, N. van der Weide, Niels, *Bicategories in univalent foundations*, 10th ed., Mathematical Structures in Computer Science, vol. 31, Cambridge University Press, 2021.

[5] N. van der Weide, Univalent Enriched Categories and the Enriched Rezk Completion, preprint arXiv:2401.11752, 2024.

A monotone-light factorization for n-categories

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Abstract.
Starting with a symmetric monoidal adjunction with certain properties, we derive another symmetric monoidal adjunction with the same properties between the respective categories of all $\mathcal{V}$-categories. If we begin with a reflection of a full replete subcategory, the derived adjunction is also a reflection of the same kind. Semi-left-exactness (also called admissibility in categorical Galois theory) or the stronger stable units property is inherited by the derived reflection. Applying these results, we conclude that the reflection of the category of all n-categories into the category of n-preorders has stable units. Then, it is also shown that this reflection determines a monotone-light factorization system on n-categories, $n \geq 1$, and that the light morphisms are precisely the n-functors faithful with respect to n-cells. In order to achieve such results, it was also shown that n-functors surjective both on vertically composable triples of horizontally composable pairs of n-cells, and on horizontally composable triples of vertically composable pairs of n-cells, are effective descent morphisms in the category of all n-categories $nCat$, $n \geq 1$.

References

[1] Carboni, A., Janelidze, G., Kelly, G. M., Paré, R. *On localization and stabilization for factorization systems.* App. Cat. Struct. **5**, (1997) 1–58.

[2] Janelidze, G., Sobral, M., Tholen, W. *Beyond Barr Exactness: Effective Descent Morphisms* in Categorical Foundations. Special Topics in Order, Topology, Algebra and Sheaf Theory, Cambridge University Press, 2004.

[3] Kelly, G. M. *Basic Concepts of Enriched Category Theory*, Reprints in Theory and Applications of Categories, No. 10, 2005.

[4] Xarez, J. J. The monotone-light factorization for categories via preorders. *Galois theory, Hopf algebras and semiabelian Categories*, 533–541, Fields Inst. Commun., 43, Amer. Math. Soc., Providence, RI, 2004.

[5] Xarez, J. J. *The monotone-light factorization for 2-categories via 2-preorders*, Theory Appl. Categories 38 (2022) 1209–1226.

[6] Xarez, J. J. *The monotone-light factorization for n-categories via n-preorders*, https://doi.org/10.48550/arXiv.2310.10475.

Toposes vs Localic Groupoids:
A unified treatment of covering theorems

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Abstract. The idea of toposes as spaces whose points have non-trivial automorphisms is said to have originated with Grothendieck, and was first made concrete by Joyal–Tierney [4]. They showed that every Grothendieck topos $\mathcal{E}$ admits a localic groupoid $\mathcal{L}(\mathcal{E})$ presenting $\mathcal{E}$, in the sense that the topos of equivariant sheaves on $\mathcal{L}(\mathcal{E})$ is equivalent to $\mathcal{E}$. This idea was then revisited by many authors (Moerdijk [5], Butz–Moerdijk [3], Awodey–Forssell [1]); there are nowadays plenty of variations on this theme in the literature. These covering theorems, or viewed another way, reconstruction theorems depend on different assumptions and are rarely directly comparable, but their proofs turn out to follow a general pattern.

In this talk, we abstract that pattern and thus reduce such a reconstruction theorem to its bare minimum; we call the minimal data associated to such a theorem its *amorphous sheaf*. These are very concrete objects: a locale and a sheaf over it satisfying certain properties. We will explain how to recover a reconstruction theorem from an amorphous sheaf, recalling the necessary amount of descent theory along the way.

Due to their tangible nature, amorphous sheaves can readily be studied from a logical point of view. For instance, we provide a logical recognition criterion for amorphous sheaves; this relies on the theory of classifying toposes for first-order theories of Butz–Johnstone [2]. Finally, we illustrate the resulting framework by reviewing a selection of established reconstruction theorems.

This is joint work with Ivan Di Liberti and Peter LeFanu Lumsdaine.

References

[1] Steve Awodey and Henrik Forssell. “First-order logical duality”. In: *Annals of Pure and Applied Logic* 164.3 (2013), pp. 319–348. DOI: 10.1016/j.apal.2012.10.016.

[2] Carsten Butz and Peter Johnstone. “Classifying toposes for first-order theories”. In: *Ann. Pure Appl. Logic* 91.1 (1998), pp. 33–58. DOI: 10.1016/S0168-0072(97)00042-0.

[3] Carsten Butz and Ieke Moerdijk. “Representing topoi by topological groupoids”. In: *J. Pure Appl. Algebra* 130.3 (1998), pp. 223–235. DOI: 10.1016/S0022-4049(97)00107-2.

[4] André Joyal and Myles Tierney. “An extension of the Galois theory of Grothendieck”. In: *Mem. Amer. Math. Soc.* 51.309 (1984), pp. vii+71. DOI: 10.1090/memo/0309.

[5] Ieke Moerdijk. “The Classifying Topos of a Continuous Groupoid. I”. In: *Transactions of the American Mathematical Society* 310.2 (1988), pp. 629–668. DOI: 10.2307/2000984.

Pivotality, twisted centres, and the anti-double of a Hopf monad

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Abstract.

Finite-dimensional Hopf algebras admit a correspondence between so-called pairs in involution, one-dimensional anti-Yetter–Drinfeld modules, and algebra isomorphisms between the Drinfeld and anti-Drinfeld double. From the perspective of representation theory, Hopf algebras are in one-to-one correspondence with rigid monoidal categories. This fact may be "categorified", passing from Hopf algebras to Hopf monads as defined by Bruguiéres and Virelizier [1]. Further, as studied by Aguiar and Chase [2], a Hopf monad may admit a comodule monad over it; this generalises the notion of a comodule algebra over a Hopf algebra, which representation theoretically expresses itself as a module category over the (rigid) monoidal base category.

In this talk, we will explore the classical theorem from this perspective, and extend it to comodule monads over Hopf monads. Hereto we construct the anti-Drinfeld double of a Hopf monad, which—analogously to the Hopf algebraic case—is a comodule over its double; the latter was studied in [3]. As it turns out the interplay between double and anti-double characterises when a rigid monoidal category is pivotal—i.e., the double dualising functor is (isomorphic to) the identity.

This talk is based on [4].

References

[1] A. Bruguières and A. Virelizier, *Hopf monads*, Adv. Math. 215, No. 2, 2007, 679–733.

[2] M. Aguiar and S. U. Chase, *Generalized Hopf modules for bimonads*, Theory Appl. Categ., Vol. 27, 2012, 263–326.

[3] A. Bruguiéres and A. Virelizier, *Quantum double of Hopf monads and categorical centers*, Trans. Am. Math. Soc. 364, No. 3, 2012, 1225–1279.

[4] S. Halbig and T. Zorman, *Pivotality, twisted centres, and the anti-double of a Hopf monad*, Theory Appl. Categ., Vol. 41, 2024, No. 4, 86–149.

Introduction to Stratified Toposes

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Abstract.
The notion of (elementary) topos abstracts to the level of categorical algebra several aspects of the category of sets. However, it is natural to assume the existence in the category of sets of Grothendieck universes, which is not reflected in the topos axioms.

In remedy of this, various notions of universe in a topos have been introduced. [1]'s axioms are already quite close to the present approach. The axioms of [5] are stronger than [1]'s, and draw on previous work on the semantics of the calculus of constructions, *e.g.* [2, 4]. We introduce a notion of universe in a topos that somewhat strengthens [5]'s axioms.

In our notion of universe, we will ask for a full logical inclusion of toposes

$$\mathcal{E} \hookrightarrow \mathcal{F} \quad ,$$

such that $\mathcal{F}$ admits an internal category $\mathfrak{U} \in \mathbf{Cat}(\mathcal{F})$ that 'represents' $\mathcal{E}$, in the sense that

$$\mathrm{Hom}_{\mathbf{Cat}(\mathcal{F})}(I, \mathfrak{U}) \simeq (\mathcal{F}/I)_{<\mathcal{E}} \quad ,$$

pseudonaturally in $I \in \mathcal{F}$, where $(\mathcal{F}/I)_{<\mathcal{E}}$ is the full subcategory of $\mathcal{F}/I$ on the maps with '$\mathcal{E}$-small fibers.' More intuitively, we then have, in particular, that

$$\mathrm{Hom}_{\mathbf{Cat}(\mathcal{F})}(I, \mathfrak{U}) \simeq \mathcal{E}/I \quad ,$$

pseudonaturally in $I \in \mathcal{E}$.

We will also ask for a novel density condition on the logical inclusion

$$\mathcal{E} \overset{i}{\hookrightarrow} \mathcal{F} \quad ,$$

which is used to extend the induced $\mathcal{E}$-indexed logical inclusion

$$\mathcal{E}/(-) \hookrightarrow i^*\mathcal{F}/(-)$$

to the expected $\mathcal{F}$-indexed logical inclusion

$$(\mathcal{F}/(-))_{<\mathcal{E}} \hookrightarrow \mathcal{F}/(-) \quad .$$

We introduce **stratified toposes**, which are toposes that admit a hierarchy of universes in our sense.[1] Whereas, in an ordinary topos, monomorphisms are 'represented' by the subobject classifier, in a stratified topos, *all morphisms* are, moreover, 'represented' by some universe.

[1] The term 'stratified topos' recalls the notion of stratified pseudotopos of [3], though the details of our proposal are closer to [5].

Key results about toposes can be refined to yield results about stratified toposes. As proof of concept, we construct what we call the **stratified topos of coalgebras** for a stratified Cartesian comonad on a stratified topos. This construction refines that of the topos of coalgebras for a Cartesian comonad on a topos. It also solves in our setting a problem that was left open by [3] in the setting of stratified pseudotoposes, and originally solved in [5]'s setting in the dissertation of the author [6].

References

[1] J. Bénabou, *Problèmes dans les Topos*, Univ. Cath. de Louvain, Inst. de Math. Pure et Appliquée, Rapport 34, 1973.

[2] J. Hyland and A. Pitts, *The theory of constructions: Categorical semantics and topos-theoretic models*, Contemporary Mathematics, vol. 92 (1989), pp. 137–199.

[3] I. Moerdijk and E. Palmgren, *Type Theories, Toposes and Constructive Set Theory: Predicative Aspects of AST*, Annals of Pure and Applied Logic, vol. 114 (2002), no. 1-3, pp. 155–201.

[4] T. Streicher, *Semantics of Type Theory*, Birkhäuser, 1991.

[5] T. Streicher, *Universes in Toposes*, From Sets and Types to Topology and Analysis: Towards Practicable Foundations for Constructive Mathematics, Oxford Logic Guides, no. 48 (2005), pp. 78–90.

[6] C. Zwanziger, *The Natural Display Topos of Coalgebras*, Ph.D. thesis, Carnegie Mellon University, 2023.